

J Fourier Anal Appl (2015) 21:32–75  
DOI 10.1007/s00041-014-9360-3

---

# Fourier Series and Twisted $C^*$ -Crossed Products

Erik Bédos · Roberto Conti

Received: 15 October 2013 / Revised: 26 June 2014 / Published online: 7 October 2014  
© Springer Science+Business Media New York 2014

**Abstract** This paper is an invitation to Fourier analysis in the context of reduced twisted  $C^*$ -crossed products associated with discrete unital twisted  $C^*$ -dynamical systems. We discuss norm-convergence of Fourier series, multipliers and summation processes. Our study relies in an essential way on the (covariant and equivariant) representation theory of  $C^*$ -dynamical systems on Hilbert  $C^*$ -modules. It also yields some information on the ideal structure of reduced twisted  $C^*$ -crossed products.

**Keywords** Fourier series · Reduced twisted  $C^*$ -crossed product · Decay properties · Multipliers · Equivariant representations · Fejér summation · Abel–Poisson summation · Invariant ideals

**Mathematics Subject Classification** 46L55 · 43A50 · 43A55

## 1 Introduction

Since its birth about two centuries ago, the theory of Fourier series has been applied to a seemingly endless number of different situations and, accordingly, it has been the subject of intensive studies, notably in relation to various kinds of convergence and

---

Communicated by Karlheinz Gröchenig.

---

E. Bédos (✉)

Institute of Mathematics, University of Oslo, Blindern, P.B. 1053, 0316 Oslo, Norway  
e-mail: bedos@math.uio.no

R. Conti

Dipartimento di Scienze di Base e Applicate per l'Ingegneria, Sez. di Matematica, Università Sapienza di Roma, via A. Scarpa 16, 00166 Roma, Italy  
e-mail: roberto.conti@sba.uniroma1.it

summation techniques. Among many others, the problem of determining conditions under which the Fourier series of a continuous periodic function on the real line is uniformly convergent has received a good deal of attention in the literature, and various kinds of summation processes have also been constructed.

In the theory of operator algebras, started in the seminal work of Murray and von Neumann, it is well known that one may associate to any group several interesting examples of  $C^*$ -algebras and von Neumann algebras. In the context of twisted group  $C^*$ -algebras (and von Neumann algebras) associated with discrete groups, the Fourier series of any element makes perfect sense. In the  $C^*$ -algebraic case, the study of norm-convergence and summation processes is more involved than in the classical set-up, but a surprisingly detailed analysis is possible, as exposed for instance in our previous article [4]. Now, given a twisted action  $(\alpha, \sigma)$  of a discrete group  $G$  on a unital  $C^*$ -algebra  $A$  (the case we discuss in this paper), one may also consider the Fourier series of any element in the so-called (reduced) crossed product  $C^*$ -algebra  $C_r^*(\Sigma)$ , where  $\Sigma$  denotes the quadruple  $(A, G, \alpha, \sigma)$ . However, the Fourier coefficients lie now in  $A$  (rather than in  $\mathbb{C}$ ), and consequently, the analysis becomes much more challenging. Our main aim with the present work is to investigate how one can handle this situation, having in mind the wild variety of different cases that may appear. Due to the success of the theory of classical Fourier series, we expect that once put on solid grounds the corresponding theory will become a useful tool in the study of  $C^*$ -dynamical systems.

We stress that the idea of doing Fourier analysis in reduced  $C^*$ -crossed products is not new. It is already present in Zeller-Meier's impressive article [48] from 1968, where he, among many other results, shows the existence of summation processes in the case of amenable groups, and uses this to obtain some valuable information about the ideal structure. In the book by Davidson [13] one can find a proof of the direct analogue of the Fejér summation process in the special case of crossed products by an action of  $\mathbb{Z}$  (see also [45]). One may also consider  $C_r^*(\Sigma)$  as the reduced cross sectional algebra of a Fell bundle over the discrete group  $G$  (see [22]) and notice that Exel [19] has shown how to associate Fourier series to elements in such algebras. In the same paper, Exel constructs summation processes in the case of Fell bundles with the so-called approximation property and illustrates their usefulness when studying induced ideals in the sectional algebra (see also [20]). One could therefore think that one may as well work in the more general setting of Fell bundles. However, as alluded to in [5], our attitude has been that it should be possible to develop a more powerful analysis by exploiting the structure of discrete twisted  $C^*$ -crossed products and their representation theory on Hilbert  $C^*$ -modules. We will do our best to justify this point of view and add some further evidence to the fact that the equivariant representations of  $\Sigma$  on Hilbert  $A$ -modules introduced in [5] play a role complementary to the one played by covariant representations. Note that when  $A$  is trivial, this splitting is not visible: covariant and equivariant representations coincide in this case and amount to unitary representations of  $G$ .

The starting point for our approach is as follows. As is well known (see e.g. [5, 48]),  $B = C_r^*(\Sigma)$  may be characterized (up to isomorphism) as a  $C^*$ -algebra  $B$  that is generated by a copy of  $A$  and a family of unitaries  $\{u_g\}_{g \in G}$  satisfying the relations  $u_g a = \alpha_g(a) u_g$  and  $u_g u_h = \sigma(g, h) u_{gh}$ , and is equipped with a faithful conditional expectation  $E$  from  $B$  onto  $A$  satisfying  $E(u_g) = 0$  when  $g \neq e$  (the identity of

$G$ ). The expectation  $E$  may be thought as some kind of  $A$ -valued Haar integral: if  $G$  is abelian and  $\alpha, \sigma$  are both trivial, then  $C_r^*(\Sigma)$  is isomorphic to  $B = C(\widehat{G}, A)$ , the continuous functions on the dual group  $\widehat{G}$  with values in  $A$ , and  $E$  is indeed given by the  $A$ -valued integral  $E(f) = \int_{\widehat{G}} f(\gamma) d\gamma$  with respect to the normalized Haar measure on  $\widehat{G}$ . The Fourier coefficients of  $x \in C_r^*(\Sigma)$  are therefore usually defined by setting  $\widehat{x}(g) = E(xu_g^*)$ , so the Fourier transform  $\widehat{x}$  becomes a function from  $G$  to  $A$ . A useful fact that is not immediately apparent from this definition is that  $\widehat{x}$  lies in the space

$$H^\Sigma = \left\{ \xi : G \rightarrow A \mid \sum_{g \in G} \alpha_g^{-1}(\xi(g)^* \xi(g)) \text{ is norm-convergent in } A \right\}.$$

Note that this statement contains nontrivial information about the decay at infinity of general Fourier coefficients (Riemann–Lebesgue Lemma). In the case of ordinary reduced crossed products this was recently observed in [42]. However, as was already remarked by Anantharaman-Delaroche in [1], the space  $H^\Sigma$  has a natural Hilbert  $A$ -module structure on which  $C_r^*(\Sigma)$  may be faithfully represented by adjointable operators. The Fourier transform of  $x \in C_r^*(\Sigma)$  is then simply defined by  $\widehat{x} = x\xi_0 \in H^\Sigma$ , where  $\xi_0(g) = \delta_{g,e}1$ , while  $E$  is given on  $C_r^*(\Sigma)$  by  $E(x) = \widehat{x}(e)$ . The (formal) Fourier series of  $x$  is now defined as

$$\sum_{g \in G} \widehat{x}(g) \lambda_\Sigma(g)$$

where the  $\lambda_\Sigma(g)$ 's denote the canonical unitaries of  $C_r^*(\Sigma)$  when it acts on  $H^\Sigma$ .

Following [4], the general problem about norm-convergence of Fourier series in  $C_r^*(\Sigma)$  may be considered as the search for “decay subspaces” of  $H^\Sigma$  that are as large as possible. For example,  $\ell^1(G, A)$  is a decay subspace, corresponding to elements of  $C_r^*(\Sigma)$  with absolutely convergent Fourier series. Moreover, inspired by Jolissaint’s notion of rapid decay (RD) for groups [32], any weight function  $\kappa : G \rightarrow [1, \infty)$  such that  $G$  is  $\kappa$ -decaying in the sense of [4] leads to a decay subspace  $\ell_\kappa^2(G, A)$  of  $H^\Sigma$ . However, it appears that a larger subspace  $H_\kappa^\Sigma$  is needed if one wishes to follow the strategy pioneered by Haagerup [26] and establish the existence of certain summation processes, as we did in [4] for many nonamenable groups. A problem that appears naturally in our framework is thus to find conditions ensuring that  $\Sigma$  has the  $H_\kappa^\Sigma$ -decay property (expressing that  $H_\kappa^\Sigma$  is a “decay subspace”), again a kind of generalized version with coefficients of the RD-property. When  $A$  is commutative and  $\alpha$  is trivial, it suffices to assume that  $G$  is  $\kappa$ -decaying (we prove this in the final section), but it should be possible to relax these assumptions. Ji and Schweitzer [31] have a result in this direction for nontrivial actions (and groups of polynomial growth), but it is not obvious to us that their proof can be adapted to give a more general result.

We next focus on (reduced) multipliers of  $\Sigma$ . Our ultimate goal in this paper is to use such maps as smoothing kernels for summation processes for Fourier series, as we did in [4] when  $A = \mathbb{C}$ . Multipliers of  $C^*$ -dynamical systems are defined in analogy with multipliers on groups, but the terminology “multiplier” might in fact be

somewhat misleading. A multiplier  $T$  of  $\Sigma$  consists of a family  $T = \{T_g\}_{g \in G}$  of linear maps from  $A$  into itself such that there exists a bounded linear map  $M_T$  from  $C_r^*(\Sigma)$  into itself satisfying

$$\widehat{M_T(x)}(g) = T_g(\widehat{x}(g))$$

for all  $x \in C_r^*(\Sigma)$  and  $g \in G$ . Such a multiplier is called a cb-multiplier when the map  $M_T$  is completely bounded. Now, given a function  $\varphi : G \rightarrow A$ , one may wonder when it induces a “left” multiplier and consider the family  $L^\varphi = \{L_g^\varphi\}_{g \in G}$  of maps from  $A$  to itself given by  $L_g^\varphi(a) = \varphi(g)a$ . We give a set of sufficient conditions ensuring that  $L^\varphi$  is a cb-multiplier of  $\Sigma$ , and use this to show that every cb-multiplier of  $G$  induces a cb-multiplier of  $\Sigma$ . We also show that a controlled growth of  $\varphi$  w.r.t. a weight  $\kappa$ , in combination with the  $H_\kappa^\Sigma$ -decay property, suffices for  $L^\varphi$  to be a multiplier of  $\Sigma$ . In another direction, we give a conceptually satisfactory way of producing cb-multipliers of  $\Sigma$ , analogous to how matrix coefficients of unitary representations of  $G$  induce cb-multipliers on  $G$ : the maps  $T_g : A \rightarrow A$  are now of the form

$$T_g(a) = \langle x, \rho(a)v(g)y \rangle$$

for some equivariant representation  $(\rho, v)$  of  $\Sigma$  on some Hilbert  $A$ -module  $X$  and  $x, y \in X$ . The collection of these cb-multipliers on  $\Sigma$  may therefore be thought of as the analogue of the Fourier–Stieltjes algebra of  $G$ . Our proof relies on a new version of Fell’s absorption property, that loosely says that any equivariant representation of  $\Sigma$  is absorbed when tensoring with some regular covariant representation. (We use here the notion of tensor product introduced in [5]). This version complements the one proven in [5] about absorption of covariant representations when tensoring with induced regular equivariant representations.

After the study of multipliers, we turn our attention to summation processes. A Fourier summing net for  $\Sigma$  is a net  $\{T^i\}$  of multipliers of  $\Sigma$  such that, for each  $x \in C_r^*(\Sigma)$ , the Fourier series of  $M_{T^i}(x)$

$$\sum_{g \in G} T_g^i(\widehat{x}(g))\lambda_\Sigma(g)$$

is norm-convergent (necessarily to  $M_{T^i}(x)$ ) for each  $i$ , and  $M_{T^i}(x)$  converges in norm to  $x$ . The existence of such Fourier summing nets is then discussed in a number of situations. In particular, we obtain a generalization of the classical Fejér summation theorem whenever  $\Sigma$  has the weak approximation property of [5], and prove some analogs of the Abel–Poisson summation theorem. Almost all the Fourier summing nets we are effectively able to construct have the property that they preserve the invariant ideals of  $A$ . The existence of such a net affects the ideal structure of  $C_r^*(\Sigma)$ :  $\Sigma$  is then necessarily exact (in the sense of [43]), and the ideals of  $C_r^*(\Sigma)$  that are  $E$ -invariant are precisely those that are induced from invariant ideals of  $A$ . Hence, in such a situation, the problem of determining the ideals of  $C_r^*(\Sigma)$  reduces to two different tasks: finding the invariant ideals of  $A$  and investigating the possible existence of ideals of  $C_r^*(\Sigma)$  that are not  $E$ -invariant. In particular, one can then deduce that all

ideals of  $C_r^*(\Sigma)$  are induced from invariant ideals of  $A$  if one can show that any ideal of  $C_r^*(\Sigma)$  is automatically  $E$ -invariant, thereby providing an alternative approach to the one obtained in [43]. We illustrate this with a simple example in the final section.

After having described what we have done in this paper, we would like to add that we see it as a first attempt to put some facts and ideas into a wider perspective. Many issues remain to be addressed, and it might be easier to deal with some of them in specific situations (depending on various choices of the group, the algebra, the action and the cocycle) before attacking the general case. We hope that our work will stimulate further research on this topic and the reader will find many open questions and problems scattered throughout the text.

The paper is organized as follows. In Sect. 2 we collect some notions and facts from [5]. As the present article is also heavily influenced by the line of thought presented in [4], the reader is kindly advised to have a look at both these articles. Section 3 is devoted to establishing a first set of results about convergence of Fourier series. The concept of (reduced) multipliers is introduced and discussed in Sect. 4. Summation processes for Fourier series is the subject of Sect. 5. In the last section (Sect. 6) we deal with the “almost trivial” but still interesting case where  $A$  is commutative and  $\alpha$  is trivial, and show that in this situation the cocycle does not create any trouble for the analysis.

## 2 Preliminaries

Throughout the paper, we will use the following conventions. To avoid some technicalities, we will only work in the category of *unital*  $C^*$ -algebras, and a homomorphism between two objects in this category will always mean a unit preserving  $*$ -homomorphism. Isomorphisms and automorphisms are consequently also assumed to be  $*$ -preserving. The group of unitary elements in a  $C^*$ -algebra  $A$  will be denoted by  $\mathcal{U}(A)$ , the center of  $A$  by  $Z(A)$ , while the group of automorphisms of  $A$  will be denoted by  $\text{Aut}(A)$ . The identity map on  $A$  will be denoted by  $\text{id}$  (or  $\text{id}_A$ ). By an ideal of  $A$ , we will always mean a two-sided closed ideal, unless otherwise specified. If  $B$  is another  $C^*$ -algebra,  $A \otimes B$  will denote their minimal tensor product.

By a Hilbert  $C^*$ -module, we will always mean a *right* Hilbert  $C^*$ -module and follow the notation introduced in [35]. In particular, all inner products will be assumed to be linear in the second variable,  $\mathcal{L}(X, Y)$  will denote the space of all adjointable operators between two Hilbert  $C^*$ -modules  $X$  and  $Y$  over a  $C^*$ -algebra  $B$ , and  $\mathcal{L}(X) = \mathcal{L}(X, X)$ . A representation of a  $C^*$ -algebra  $A$  on a Hilbert  $B$ -module  $Y$  is then a homomorphism from  $A$  into the  $C^*$ -algebra  $\mathcal{L}(Y)$ . If  $Z$  is another Hilbert  $C^*$ -module (over  $C$ ), we will let  $\pi \otimes \iota : A \rightarrow \mathcal{L}(Y \otimes Z)$  denote the amplified representation of  $A$  on  $Y \otimes Z$  given by  $(\pi \otimes \iota)(a) = \pi(a) \otimes I_Z$ , where the Hilbert  $B \otimes C$ -module  $Y \otimes Z$  is the external tensor product of  $Y$  and  $Z$  and  $I_Z$  denotes the identity operator on  $Z$ . Note that if  $Z$  is a Hilbert space, i.e., a Hilbert  $\mathbb{C}$ -module, then we may and will regard  $Y \otimes Z$  as a Hilbert  $B$ -module.

We will work with series in a  $C^*$ -algebra  $A$  of the form  $\sum_{i \in I} a_i$  where  $I$  is a possibly uncountable set and  $a_i \in A$  for each  $i \in I$ . Norm-convergence of such a series will always mean unconditional convergence (sometimes called summability).

Since  $A$  is a Banach space, this happens if and only if the usual Cauchy criterion is satisfied [14]. An immediate consequence is the following fact, used without notice on several occasions in the sequel: if  $\{a_i\}_{i \in I}, \{b_i\}_{i \in I}$  are families of elements in  $A^+$  (the cone of positive elements in  $A$ ) such that  $a_i \leq b_i$  for each  $i \in I$  and  $\sum_{i \in I} b_i$  is norm-convergent to  $b$  (lying necessarily in  $A^+$ ), then  $\sum_{i \in I} a_i$  is norm-convergent to some  $a$  in  $A^+$  satisfying  $a \leq b$  (hence also  $\|a\| \leq \|b\|$ ).

The quadruple  $\Sigma = (A, G, \alpha, \sigma)$  will always denote a *twisted* (unital, discrete)  $C^*$ -dynamical system. This means that  $A$  is a  $C^*$ -algebra with unit 1,  $G$  is a discrete group with identity  $e$  and  $(\alpha, \sigma)$  is a *twisted action* of  $G$  on  $A$  (sometimes called a cocycle  $G$ -action on  $A$ ), that is,  $\alpha$  is a map from  $G$  into  $\text{Aut}(A)$  and  $\sigma$  is a map from  $G \times G$  into  $\mathcal{U}(A)$ , satisfying

$$\begin{aligned}\alpha_g \alpha_h &= \text{Ad}(\sigma(g, h)) \alpha_{gh} \\ \sigma(g, h) \sigma(gh, k) &= \alpha_g(\sigma(h, k)) \sigma(g, hk) \\ \sigma(g, e) &= \sigma(e, g) = 1,\end{aligned}$$

for all  $g, h, k \in G$ . Of course,  $\text{Ad}(v)$  denotes here the (inner) automorphism of  $A$  implemented by some unitary  $v$  in  $\mathcal{U}(A)$ .

If  $\sigma$  is trivial, that is,  $\sigma(g, h) = 1$  for all  $g, h \in G$ , then  $\Sigma$  is an ordinary  $C^*$ -dynamical system (see e.g. [9, 15, 47]), and one just writes  $\Sigma = (A, G, \alpha)$ . If  $\sigma$  is *central*, that is, takes values in  $\mathcal{U}(Z(A))$ , then  $\alpha$  is an ordinary action of  $G$  on  $A$ , and this is the case studied in [48]. If  $A = \mathbb{C}$ , then  $\alpha_g = \text{id}$  for all  $g \in G$  and  $\sigma$  is a 2-cocycle on  $G$  with values in the unit circle  $\mathbb{T}$ , (see e.g. [4] and references therein).

To each twisted  $C^*$ -dynamical system  $\Sigma = (A, G, \alpha, \sigma)$  one may associate its *full twisted crossed product*  $C^*(\Sigma)$  and its *reduced* version  $C_r^*(\Sigma)$  (see [38, 39]). In this paper we will be mostly interested in the reduced algebra. For the ease of the reader, we will recall some definitions and facts from [5] needed in the sequel.

A *covariant homomorphism* of  $\Sigma$  is a pair  $(\pi, u)$ , where  $\pi$  is a homomorphism of  $A$  into a  $C^*$ -algebra  $C$  and  $u$  is a map of  $G$  into  $\mathcal{U}(C)$ , which satisfy

$$u(g)u(h) = \pi(\sigma(g, h))u(gh)$$

and the covariance relation

$$\pi(\alpha_g(a)) = u(g)\pi(a)u(g)^* \quad (1)$$

for all  $g, h \in G$  and  $a \in A$ . Every such pair induces a unique canonical homomorphism  $\pi \times u$  from  $C^*(\Sigma)$  onto the  $C^*$ -subalgebra of  $C$  generated by  $\pi(A)$  and  $u(G)$ . If  $C = \mathcal{L}(X)$  for some Hilbert  $C^*$ -module  $X$ , then  $(\pi, u)$  is called a *covariant representation* of  $\Sigma$  on  $X$ .

Let  $Y$  be a Hilbert  $B$ -module and assume  $\pi$  is a representation of  $A$  on  $Y$ . We can then form the Hilbert  $B$ -module  $H_Y^G (\simeq Y \otimes \ell^2(G))$  given by

$$H_Y^G = \left\{ \xi : G \rightarrow Y \mid \sum_{g \in G} \langle \xi(g), \xi(g) \rangle \text{ is norm-convergent in } B \right\} \quad (2)$$

endowed with the  $B$ -valued inner product

$$\langle \xi, \eta \rangle = \sum_{g \in G} \langle \xi(g), \eta(g) \rangle, \quad \text{for all } \xi, \eta \in H_Y^G,$$

and the natural module right action of  $B$  given by

$$(\xi \cdot b)(g) = \xi(g)b \text{ for all } \xi \in H_Y^G, b \in B \text{ and } g \in G.$$

The *regular covariant representation*  $(\tilde{\pi}, \tilde{\lambda}_\pi)$  of  $\Sigma$  on  $H_Y^G$  associated to  $\pi$  is then defined by

$$(\tilde{\pi}(a)\xi)(h) = \pi(\alpha_h^{-1}(a))\xi(h), \text{ for all } a \in A, \xi \in H_Y^G \text{ and } h \in G, \quad (3)$$

$$(\tilde{\lambda}_\pi(g)\xi)(h) = \pi(\alpha_h^{-1}(\sigma(g, g^{-1}h)))\xi(g^{-1}h), \text{ for all } g, h \in G \text{ and } \xi \in H_Y^G. \quad (4)$$

Considering  $A$  as a Hilbert  $A$ -module in the standard way and letting  $\ell : A \rightarrow \mathcal{L}(A)$  be given by  $\ell(a)(a') = aa'$ , for  $a, a' \in A$ , we get the *regular covariant representation*  $(\tilde{\ell}, \tilde{\lambda}_\ell)$  associated to  $\ell$ , that acts on the Hilbert  $A$ -module

$$H_A^G = \left\{ \xi : G \rightarrow A \mid \sum_{g \in G} \xi(g)^* \xi(g) \text{ is norm-convergent in } A \right\} \quad (5)$$

in the following way:

$$(\tilde{\ell}(a)\xi)(h) = \alpha_h^{-1}(a)\xi(h), \text{ for all } a \in A, \xi \in H_A^G \text{ and } h \in G, \quad (6)$$

$$(\tilde{\lambda}_\ell(g)\xi)(h) = \alpha_h^{-1}(\sigma(g, g^{-1}h))\xi(g^{-1}h), \text{ for all } g, h \in G \text{ and } \xi \in H_A^G. \quad (7)$$

The *reduced twisted crossed product*  $C_r^*(\Sigma)$  is defined as the  $C^*$ -subalgebra of  $\mathcal{L}(H_A^G)$  generated by  $\tilde{\ell}(A)$  and  $\tilde{\lambda}_\ell(G)$ .

Setting  $\Lambda = \tilde{\ell} \times \tilde{\lambda}_\ell$ , we have  $C_r^*(\Sigma) = \Lambda(C^*(\Sigma))$ . Moreover,  $C_r^*(\Sigma) \simeq (\tilde{\pi} \times \tilde{\lambda}_\pi)(C^*(\Sigma))$  whenever  $\pi : A \rightarrow \mathcal{L}(Y)$  is a faithful representation of  $A$  on any Hilbert  $C^*$ -module  $Y$  (e.g. a Hilbert space).

It turns out to be useful to also consider the Hilbert  $A$ -module

$$H^\Sigma = \left\{ \xi : G \rightarrow A \mid \sum_{g \in G} \alpha_g^{-1}(\xi(g)^* \xi(g)) \text{ is norm-convergent in } A \right\},$$

where the right action of  $A$  on  $H^\Sigma$  and the  $A$ -valued scalar product are defined by

$$(\xi \times a)(g) = \xi(g) \alpha_g(a),$$

$$\langle \xi, \eta \rangle_\alpha = \sum_{g \in G} \alpha_g^{-1}(\xi(g)^* \eta(g)),$$

the associated norm on  $\mathcal{H}^\Sigma$  being given by  $\|\xi\|_\alpha = \left\| \sum_{g \in G} \alpha_g^{-1} (\xi(g)^* \xi(g)) \right\|^{1/2}$ .

As  $H_A^G$  and  $H^\Sigma$  are unitarily equivalent via the unitary operator  $J : H_A^G \rightarrow H^\Sigma$  given by

$$(J\eta)(g) = \alpha_g(\eta(g)), \quad \text{for all } \eta \in H_A^G \text{ and } g \in G,$$

we get a covariant representation  $(\ell_\Sigma, \lambda_\Sigma)$  of  $\Sigma$  on  $H^\Sigma$  given by

$$\ell_\Sigma(a) = J \tilde{\ell}(a) J^*, \quad \lambda_\Sigma(g) = J \tilde{\lambda}_\ell(g) J^*,$$

that is,

$$\begin{aligned} (\ell_\Sigma(a)\xi)(h) &= a\xi(h), \\ (\lambda_\Sigma(g)\xi)(h) &= \alpha_g(\xi(g^{-1}h))\sigma(g, g^{-1}h), \end{aligned}$$

for all  $a \in A$ ,  $\xi \in H^\Sigma$  and  $g, h \in G$ .

As  $\Lambda_\Sigma = \ell_\Sigma \times \lambda_\Sigma$  is unitarily equivalent to  $\Lambda$ , we may identify  $C_r^*(\Sigma)$  with  $\Lambda_\Sigma(C^*(\Sigma))$ . Further, we may also identify  $A$  with  $\ell_\Sigma(A)$ , so  $A$  acts on  $H^\Sigma$  via

$$(a\xi)(h) = a\xi(h), \quad \text{for all } a \in A, \xi \in H^\Sigma \text{ and } h \in H.$$

Letting  $C_c(\Sigma)$  denote the set of functions from  $G$  into  $A$  with finite support, and identifying it with its canonical copy inside  $C^*(\Sigma)$ , we get

$$\Lambda_\Sigma(f) = \sum_{g \in \text{supp}(f)} f(g)\lambda_\Sigma(g), \quad \text{for all } f \in C_c(\Sigma).$$

In particular, letting  $a \odot \delta_g$  denote the function in  $C_c(\Sigma)$  which is 0 everywhere except at the point  $g \in G$  where it takes the value  $a \in A$ , we have

$$\Lambda_\Sigma(a \odot \delta_g) = a\lambda_\Sigma(g).$$

The *Fourier transform* is the (injective, linear) map  $x \rightarrow \widehat{x}$  from  $C_r^*(\Sigma)$  into  $H^\Sigma$  given by

$$\widehat{x} = x\xi_0$$

where  $\xi_0 = 1 \odot \delta_e \in H^\Sigma$ .

When  $f \in C_c(\Sigma)$  and  $x \in C_r^*(\Sigma)$ , we have

$$\widehat{\Lambda_\Sigma(f)} = f, \quad \|\widehat{x}\|_\infty \leq \|\widehat{x}\|_\alpha \leq \|x\|, \quad (8)$$

where  $\|\widehat{x}\|_\infty = \sup_{g \in G} \|\widehat{x}(g)\|$ .



The *canonical conditional expectation*  $E$  from  $C_r^*(\Sigma)$  onto  $A$  is given by  $E(x) = \widehat{x}(e)$ . It satisfies that  $E(\Lambda_\Sigma(f)) = f(e)$ ,  $f \in C_c(\Sigma)$ . Moreover, we have

$$E(x^*x) = \langle \widehat{x}, \widehat{x} \rangle_\alpha, \quad E(x\lambda_\Sigma(g)^*) = \widehat{x}(g), \quad E(\lambda_\Sigma(g)x\lambda_\Sigma(g)^*) = \alpha_g(E(x))$$

for all  $x \in C_r^*(\Sigma)$  and  $g \in G$ .

Another concept, slightly adapted from [16], that will be of importance to us is the following: An *equivariant representation* of  $\Sigma$  on a Hilbert  $A$ -module  $X$  is a pair  $(\rho, v)$  where  $\rho : A \rightarrow \mathcal{L}(X)$  is a representation of  $A$  on  $X$  and  $v$  is a map from  $G$  into the group  $\mathcal{I}(X)$  consisting of all  $\mathbb{C}$ -linear, invertible, bounded maps from  $X$  into itself, which satisfy:

- (i)  $\rho(\alpha_g(a)) = v(g)\rho(a)v(g)^{-1}$ , for all  $g \in G$  and  $a \in A$ ,
- (ii)  $v(g)v(h) = \text{ad}_\rho(\sigma(g, h))v(gh)$ , for all  $g, h \in G$ ,
- (iii)  $\alpha_g(\langle x, x' \rangle) = \langle v(g)x, v(g)x' \rangle$ , for all  $g \in G$  and  $x, x' \in X$ ,
- (iv)  $v(g)(x \cdot a) = (v(g)x) \cdot \alpha_g(a)$ , for all  $g \in G, x \in X$  and  $a \in A$ .

In (ii) above,  $\text{ad}_\rho(\sigma(g, h)) \in \mathcal{I}(X)$  is defined by

$$\text{ad}_\rho(\sigma(g, h))x = (\rho(\sigma(g, h))x) \cdot \sigma(g, h)^*, \quad \text{for all } g, h \in G \text{ and } x \in X.$$

The central part of  $X$  is defined by

$$Z_X = \{z \in X \mid \rho(a)z = z \cdot a \text{ for all } a \in A\}.$$

An important feature is that whenever  $(\pi, u)$  is a covariant representation of  $\Sigma$  on some Hilbert  $B$ -module  $Y$ , we can form the product covariant representation  $(\rho \otimes \pi, v \otimes u)$  of  $\Sigma$  on the Hilbert  $B$ -module  $X \otimes_\pi Y$ , see [5, Sect. 4].

The *trivial equivariant representation* of  $\Sigma$  is the pair  $(\ell, \alpha)$  acting on the  $A$ -module  $A$  (with its canonical structure). The *regular equivariant representation* of  $\Sigma$  is the pair  $(\check{\ell}, \check{\alpha})$  on  $H_A^G$  defined by

$$(\check{\ell}(a)\xi)(h) = a\xi(h), \quad (\check{\alpha}(g)\xi)(h) = \alpha_g(\xi(g^{-1}h))$$

for  $a \in A$ ,  $\xi \in H_A^G$  and  $g, h \in G$ . More generally, if  $(\rho, v)$  is an equivariant representation of  $\Sigma$  on a Hilbert  $A$ -module  $X$ , it induces an equivariant representation  $(\check{\rho}, \check{v})$  of  $\Sigma$  on  $H_X^G$  given by

$$(\check{\rho}(a)\xi)(h) = \rho(a)\xi(h), \quad (\check{v}(g)\xi)(h) = v(g)\xi(g^{-1}h),$$

for all  $a \in A$ ,  $\xi \in H_X^G$  and  $g, h \in G$ .

We recall from [5] that  $\Sigma$  is said to have the *weak approximation property* if there exist an equivariant representation  $(\rho, v)$  of  $\Sigma$  on some  $A$ -module  $X$  and nets  $\{\xi_i\}$ ,  $\{\eta_i\}$  in  $H_X^G$ , (that both may be chosen with finite support) satisfying

- (a) there exists some  $M > 0$  such that  $\|\xi_i\| \cdot \|\eta_i\| \leq M$  for all  $i$ ;

(b) for all  $g \in G$  and  $a \in A$  we have  $\lim_i \|\langle \xi_i, \check{\rho}(a)\check{v}(g)\eta_i \rangle - a\| = 0$ , i.e.,

$$\lim_i \sum_{h \in G} \langle \xi_i(h), \rho(a) v(g)\eta_i(g^{-1}h) \rangle = a.$$

As shown in [5, Theorem 5.11], the weak approximation property is enough to ensure regularity of  $\Sigma$ , that is,  $\Lambda : C^*(\Sigma) \rightarrow C_r^*(\Sigma)$  is then an isomorphism.

If  $(\rho, v)$  can be chosen to be equal to  $(\ell, \alpha)$  in the above definition, one recovers the *approximation property* introduced by Exel [19] (see also [23]).

If  $\{\xi_i\}$  or  $\{\eta_i\}$  (resp.  $\{\xi_i\}$  and  $\{\eta_i\}$ ) can be chosen to lie in the central part of  $H_X^G$ , we will say that  $\Sigma$  has the *half-central* (resp. *central*) *weak approximation property*. See Remarks 5.9 and 5.10 in [5] for a discussion of other related notions.

### 3 Convergence of Fourier Series

Given  $x \in C_r^*(\Sigma) \subset \mathcal{L}(H^\Sigma)$ , its (formal) *Fourier series* is defined by

$$\sum_{g \in G} \widehat{x}(g) \lambda_\Sigma(g).$$

It is well known that this series will not necessarily be convergent w.r.t. the operator norm  $\|\cdot\|$  on  $\mathcal{L}(H^\Sigma)$  (even in the classical case where  $A, \alpha$  and  $\sigma$  are all trivial and  $G$  is abelian).

However, if we consider the norm on  $C_r^*(\Sigma)$  given by  $\|x\|_\alpha = \|\widehat{x}\|_\alpha$ , then the Fourier series of  $x \in C_r^*(\Sigma)$  converges to  $x$  w.r.t.  $\|\cdot\|_\alpha$ .

Indeed, for  $F \subset G$ ,  $F$  finite, set  $x_F = \sum_{g \in F} \widehat{x}(g) \lambda_\Sigma(g)$ . Letting  $\chi_F$  denote the characteristic function of  $F$  in  $G$ , we have

$$\widehat{x_F}(g) = \begin{cases} \widehat{x}(g), & g \in F \\ 0, & g \notin F \end{cases}$$

for all  $g \in G$ . It follows that  $\|x_F - x\|_\alpha = \|\widehat{x}\chi_F - \widehat{x}\|_\alpha \rightarrow 0$  as  $F \uparrow G$ .

For later use we also record a related fact.

**Proposition 3.1** *Let  $\xi : G \rightarrow A$  and assume that  $\sum_{g \in G} \xi(g) \lambda_\Sigma(g)$  converges to some  $x \in C_r^*(\Sigma)$  w.r.t.  $\|\cdot\|_\alpha$ . Then  $\xi \in H^\Sigma$  and  $\xi = \widehat{x}$ .*

*Proof* Let  $F$  be a finite subset of  $G$  and set

$$y_F = \sum_{g \in F} \xi(g) \lambda_\Sigma(g).$$

Then  $\widehat{y_F} = \xi_F$  where  $\xi_F = \xi \chi_F$ . As  $y_F \rightarrow x$  w.r.t.  $\|\cdot\|_\alpha$  by assumption, we have  $\xi_F = \widehat{y_F} \rightarrow \widehat{x}$  w.r.t.  $\|\cdot\|_\alpha$ . Hence, for every  $g \in G$ , we get

$$\|(\widehat{x} - \xi_F)(g)\|^2 = \|\alpha_g^{-1}((\widehat{x} - \xi_F)(g)^*(\widehat{x} - \xi_F)(g))\| \leq \|\widehat{x} - \xi_F\|_\alpha^2 \rightarrow 0$$

as  $F \uparrow G$ . That is,  $\xi_F(g) \rightarrow \widehat{x}(g)$  in norm for every  $g \in G$ , which gives

$$\xi(g) = \lim_{F \uparrow G} \xi_F(g) = \widehat{x}(g), \quad g \in G.$$

So  $\xi = \widehat{x} \in H^\Sigma$ , as asserted.  $\square$

We set

$$CF(\Sigma) = \left\{ x \in C_r^*(\Sigma) \mid \sum_{g \in G} \widehat{x}(g) \lambda_\Sigma(g) \text{ is convergent w.r.t. } \|\cdot\| \right\}.$$

Note that if  $x \in CF(\Sigma)$ , then, as  $\|\cdot\|_\alpha \leq \|\cdot\|$ , it readily follows from what we just have seen that the Fourier series of  $x$  necessarily converges to  $x$  w.r.t.  $\|\cdot\|$ . In order to describe some subspaces of  $CF(\Sigma)$ , we adapt some definitions from [4].

Let  $\mathcal{L}$  be a subspace of  $H^\Sigma$  containing  $C_c(\Sigma)$  and let  $\|\cdot\|'$  be a norm on  $\mathcal{L}$ . If  $\xi \in \mathcal{L}$ , then we will say that  $\xi \rightarrow 0$  at infinity (w.r.t.  $\|\cdot\|'$ ) when, for every  $\epsilon > 0$ , there exists a finite subset  $F_0$  of  $G$  such that  $\|\xi_F\|' < \epsilon$  for all finite subsets  $F$  disjoint from  $F_0$ . We will also say that  $\Sigma$  has the  $\mathcal{L}$ -decay property (w.r.t.  $\|\cdot\|'$ ) if  $\xi \rightarrow 0$  at infinity for every  $\xi \in \mathcal{L}$  and there exists some  $C > 0$  such that

$$\|\Lambda_\Sigma(f)\| \leq C\|f\|' \quad \text{for all } f \in C_c(\Sigma). \quad (9)$$

**Proposition 3.2** *Let  $\Sigma$  have the  $\mathcal{L}$ -decay property (w.r.t.  $\|\cdot\|'$ ) and let  $\xi \in \mathcal{L}$ .*

*Then  $\sum_{g \in G} \xi(g) \lambda_\Sigma(g)$  converges in operator norm to some  $x \in C_r^*(\Sigma)$  satisfying  $\widehat{x} = \xi$ .*

*Denoting this  $x$  by  $\tilde{\Lambda}_\Sigma(\xi)$ , and letting  $\tilde{\Lambda}_\Sigma : \mathcal{L} \rightarrow C_r^*(\Sigma)$  be the associated map, we have*

$$\tilde{\Lambda}_\Sigma(\mathcal{L}) = \{x \in C_r^*(\Sigma) \mid \widehat{x} \in \mathcal{L}\} \subset CF(\Sigma). \quad (10)$$

*Proof* This proposition is the direct analogue of [4, Lemma 3.4] and [4, Theorem 3.5], and their proofs adapt in a verbatim way (using Proposition 3.1 instead of [4, Proposition 2.10]).  $\square$

For example, consider the space

$$\ell^1(G, A) = \left\{ \xi : G \rightarrow A \mid \sum_{g \in G} \|\xi(g)\| < \infty \right\}$$

and let  $\|\cdot\|_1$  denote the associated norm. Then, clearly,  $\ell^1(G, A)$  is a subspace of  $H^\Sigma$ . Moreover,  $\Sigma$  has the  $\ell^1(G, A)$ -decay property (w.r.t.  $\|\cdot\|_1$ ):  $\xi \rightarrow 0$  at infinity for every  $\xi \in \ell^1(G, A)$  (since this property holds in  $\ell^1(G)$ ) and

$$\|\Lambda_\Sigma(f)\| \leq \sum_{g \in \text{supp}(f)} \|f(g) \lambda_\Sigma(g)\| = \sum_{g \in \text{supp}(f)} \|f(g)\| = \|f\|_1$$

holds for every  $f \in C_c(\Sigma)$ .

The space  $\ell^2(G, A) = \{\xi : G \rightarrow A \mid \sum_{g \in G} \|\xi(g)\|^2 < \infty\}$ , equipped with its natural norm  $\|\cdot\|_2$ , is also a subspace of  $H^\Sigma$ , but it can not be expected that  $\Sigma$  will have the  $\ell^2(G, A)$ -decay property (as this is not true when  $A = \mathbb{C}$ , unless if  $G$  is finite).

We may instead consider weighted  $\ell^2$ -spaces. Dealing only with the scalar-valued case, we pick some function  $\kappa : G \rightarrow [1, +\infty)$  and equip the space

$$\ell_\kappa^2(G, A) = \left\{ \xi : G \rightarrow A \mid \sum_g \|\xi(g)\|^2 \kappa(g)^2 < +\infty \right\}$$

with its natural norm  $\|\xi\|_{2,\kappa} = \|\xi\kappa\|_2$ .

For example, assume that  $\kappa^{-1} \in \ell^2(G)$ . Then  $\Sigma$  has the  $\ell_\kappa^2(G, A)$ -decay property (w.r.t.  $\|\cdot\|_{2,\kappa}$ ). Indeed, it follows from the Cauchy-Schwarz inequality that

$$\|\Lambda_\Sigma(f)\| \leq \|f\|_1 \leq \|\kappa^{-1}\|_2 \|f\|_{2,\kappa}$$

for every  $f \in C_c(\Sigma)$ , and the assertion easily follows.

In [4], we introduced the notion of  $\kappa$ -decay for the group  $G$ . It just expresses that the system  $(\mathbb{C}, G, \text{id}, 1)$  has the  $\ell_\kappa^2(G, \mathbb{C})$ -decay property (w.r.t.  $\|\cdot\|_{2,\kappa}$ ). Note that the inequality (9) then just amounts to

$$\|f * g\|_2 \leq C \|f\|_{2,\kappa} \|g\|_2$$

for all  $f \in C_c(G)$  and  $g \in \ell^2(G)$  (where  $f * g$  denotes the usual convolution product), and the least possible  $C > 0$  is called the *decay constant*. We also note that any countable group is  $\kappa$ -decaying for a suitable choice of  $\kappa$  with relatively slow growth; see for example [4, Lemma 3.14].

**Proposition 3.3** *Suppose that  $G$  is  $\kappa$ -decaying. Then  $\Sigma$  has the  $\ell_\kappa^2(G, A)$ -decay property (w.r.t.  $\|\cdot\|_{2,\kappa}$ ). Hence, if  $x \in C_r^*(\Sigma)$  and  $\hat{x} \in \ell_\kappa^2(G, A)$ , then  $x \in CF(\Sigma)$ .*

*Proof* If  $\xi \in \ell_\kappa^2(G, A)$ , then  $\xi \rightarrow 0$  at infinity w.r.t.  $\|\cdot\|_{2,\kappa}$  (since  $g \rightarrow \|(\xi\kappa)(g)\|$  is a function in  $\ell^2(G)$  and therefore goes to 0 at infinity).

Next, we pick a faithful representation  $\pi$  of  $A$  on some Hilbert space  $\mathcal{H}$  with associated norm  $\|\cdot\|_{\mathcal{H}}$ , and form the regular representation  $\tilde{\pi} \times \tilde{\lambda}_\pi$  of  $C^*(\Sigma)$  on  $\tilde{\mathcal{H}} = \ell^2(G, \mathcal{H})$  with associated norm given by  $\|\eta\|_{\tilde{\mathcal{H}}} = \left(\sum_{g \in G} \|\eta(g)\|_{\mathcal{H}}^2\right)^{1/2}$ .

As  $\|\Lambda_\Sigma(f)\| = \|\Lambda(f)\| = \|(\tilde{\pi} \times \tilde{\lambda}_\pi)f\|$  for all  $f \in C_c(\Sigma)$ , it suffices to show that there exists some  $C > 0$  such that, for all  $f \in C_c(\Sigma)$ , we have

$$\|(\tilde{\pi} \times \tilde{\lambda}_\pi)f\| \leq C \|f\|_{2,\kappa}.$$

So let  $f \in C_c(\Sigma)$ . Set  $F = \text{supp}(f)$  and define  $\tilde{f}(g) = \|f(g)\|$ , for  $g \in G$ , so  $\tilde{f} \in C_c(G)$ . Then

$$\|f\|_{2,\kappa}^2 = \sum_{g \in F} \|f(g)\kappa(g)\|^2 = \sum_{g \in F} |\tilde{f}(g)|^2 \kappa(g)^2 = \|\tilde{f}\|_{2,\kappa}^2.$$

Let  $\xi \in \tilde{\mathcal{H}}$  and, for each  $g \in G$ , set  $\tilde{\xi}(g) = \|\xi(g)\|_{\mathcal{H}}$ , so  $\tilde{\xi} \in \ell^2(G)$  and  $\|\tilde{\xi}\|_2 = \|\xi\|_{\tilde{\mathcal{H}}}$ .  
Let now  $h \in G$ . Then we have

$$\begin{aligned} \|[(\tilde{\pi} \times \tilde{\lambda}_{\pi})f]\xi\|_{\mathcal{H}}(h) &= \left\| \left( \sum_{g \in F} \tilde{\pi}(f(g)) \tilde{\lambda}_{\pi}(g) \xi \right) (h) \right\|_{\mathcal{H}} \\ &= \left\| \sum_{g \in F} \pi(\alpha_h^{-1}(f(g))) \pi(\alpha_h^{-1}(\sigma(g, g^{-1}h))) \xi(g^{-1}h) \right\|_{\mathcal{H}} \\ &\leq \sum_{g \in F} \|\pi(\alpha_h^{-1}(f(g)))\| \|\pi(\alpha_h^{-1}(\sigma(g, g^{-1}h))) \xi(g^{-1}h)\|_{\mathcal{H}} \\ &= \sum_{g \in F} \|f(g)\| \|\xi(g^{-1}h)\|_{\mathcal{H}} = \sum_{g \in F} \tilde{f}(g) \tilde{\xi}(g^{-1}h) \\ &= (\tilde{f} * \tilde{\xi})(h). \end{aligned}$$

This gives

$$\begin{aligned} \|[(\tilde{\pi} \times \tilde{\lambda}_{\pi})f]\xi\|_{\tilde{\mathcal{H}}}^2 &\leq \sum_{h \in G} |(\tilde{f} * \tilde{\xi})(h)|^2 = \|\tilde{f} * \tilde{\xi}\|_2^2 \\ &\leq C^2 \|\tilde{f}\|_{2,\kappa}^2 \|\tilde{\xi}\|_2^2 = C^2 \|f\|_{2,\kappa}^2 \|\xi\|_{\tilde{\mathcal{H}}}^2, \end{aligned}$$

where  $C$  denotes the decay constant of  $G$  w.r.t.  $\kappa$ . Hence,

$$\|(\tilde{\pi} \times \tilde{\lambda}_{\pi})f\| \leq C \|f\|_{2,\kappa}$$

as desired. The last assertion then follows from Proposition 3.2.  $\square$

We may now obtain a generalized version of [4, Theorem 3.15]. It relies on the concept of polynomial (resp. subexponential)  $H$ -growth that we introduced in [4]. For amenable groups these concepts of growth coincide with the classical ones. For other examples, see [4, Sect. 3].

**Corollary 3.4** *Let  $K : G \rightarrow [0, \infty)$  be a proper function. If  $G$  has polynomial  $H$ -growth (w.r.t.  $K$ ), then there exists some  $s > 0$  such that the Fourier series of  $x \in C_r^*(\Sigma)$  converges to  $x$  in operator norm whenever*

$$\sum_{g \in G} \|\widehat{x}(g)\|^2 (1 + K(g))^s < \infty.$$

*If  $G$  has subexponential  $H$ -growth (w.r.t.  $K$ ), then the Fourier series of  $x \in C_r^*(\Sigma)$  converges to  $x$  in operator norm whenever there exists some  $t > 0$  such that*

$$\sum_{g \in G} \|\widehat{x}(g)\|^2 \exp(tK(g)) < \infty.$$

*Proof* Assume that  $G$  has polynomial H-growth (w.r.t.  $K$ ). By [4, Theorem 3.13, part 1], we know that  $G$  is  $(1 + K)^s$ -decaying for some  $s > 0$ . The first assertion is then a direct consequence of Proposition 3.3. If  $G$  has subexponential H-growth (w.r.t.  $K$ ), the proof is similar, except that we now use [4, Theorem 3.13, part 2].  $\square$

In some aspects (see for example Proposition 4.16 and its following remark), the spaces  $\ell_\kappa^2(G, A)$ , as subspaces of  $H^\Sigma$ , seem to be too small when  $A$  is non-trivial, and one should instead consider the subspaces

$$\begin{aligned} H_\kappa^\Sigma &= \left\{ \xi : G \rightarrow A \mid \sum_{g \in G} \alpha_g^{-1} (\xi(g)^* \xi(g)) \kappa(g)^2 \text{ is norm-convergent in } A \right\} \\ &= \{ \xi : G \rightarrow A \mid \xi \kappa \in H^\Sigma \} \end{aligned}$$

equipped with the norm  $\|\xi\|_{\alpha, \kappa} = \|\xi \kappa\|_\alpha$ . In the case where  $A$  is commutative,  $\sigma$  is scalar-valued and  $\kappa$  is of the form  $\kappa = (1 + K)^m$  for some proper length function  $K$  on  $G$  and  $m \in \mathbb{N}$ , such subspaces have previously been considered in [10, 31]. Note that we have

$$\begin{array}{ccc} \ell^1(G, A) & \subset & \ell^2(G, A) \subset H^\Sigma \\ & \cup & \cup \\ & \ell_\kappa^2(G, A) & \subset H_\kappa^\Sigma \end{array}$$

Moreover, if  $\Sigma$  has the  $H_\kappa^\Sigma$ -decay property, then  $\Sigma$  also has the  $\ell_\kappa^2(G, A)$ -decay property (as  $\|f\|_{\alpha, \kappa} \leq \|f\|_{2, \kappa}$ ,  $f \in C_c(\Sigma)$ ).

We will see in Corollary 6.3 that  $\Sigma$  has the  $H_\kappa^\Sigma$ -decay property whenever  $A$  is commutative,  $\alpha$  is trivial and  $G$  is  $\kappa$ -decaying. We expect that the  $H_\kappa^\Sigma$ -decay property will also hold in some cases where the action is not trivial, but leave this open for future investigations. We only mention that it might be useful to consider the notion of  $\Sigma$ -content, defined for a finite nonempty subset  $E$  of  $G$  by

$$C_\Sigma(E) = \sup \{ \|\Lambda_\Sigma(f)\| \mid f \in C_c(\Sigma), \text{supp}(f) \subset E, \|f\|_\alpha = 1 \}.$$

To see that  $C_\Sigma(E)$  is finite, consider  $f \in C_c(\Sigma)$  satisfying  $\text{supp}(f) \subset E$  and  $\|f\|_\alpha = 1$ . As  $\|f(g)\| \leq \|f\|_\alpha = 1$  for all  $g \in G$ , we have

$$\|\Lambda_\Sigma(f)\| \leq \sum_{g \in G} \|f(g)\| \leq |E| \|f\|_\alpha = |E|.$$

Hence,  $C_\Sigma(E) \leq |E| < \infty$ . One can also check that if  $F$  is another finite nonempty subset of  $G$ , then we have  $C_\Sigma(E) \leq C_\Sigma(F)$  whenever  $E \subset F$ , and  $C_\Sigma(E \cup F) \leq C_\Sigma(E) + C_\Sigma(F)$  whenever  $E$  and  $F$  are disjoint.

When  $A = \mathbb{C}$ , then it is not difficult to see that  $C_\Sigma(E) \leq c(E) \leq |E|^{1/2}$ , where  $c(E)$  denotes the Haagerup content of  $E$  (as defined in [4]). To proceed further, one will need to develop techniques to obtain more precise estimates for  $C_\Sigma(E)$  in the general case.

## 4 Multipliers

We will let  $MA(G)$  (resp.  $M_0A(G)$ ) denote the space of multipliers (resp. cb-multipliers) on  $G$ , as defined for example in [12, 41], and considered in [4]. We recall that

$$B(G) \subset UB(G) \subset M_0A(G) \subset MA(G)$$

where  $B(G)$  denotes the Fourier–Stieltjes algebra of  $G$  and  $UB(G)$  denotes the algebra consisting of the matrix coefficients of uniformly bounded representations of  $G$ . (It is also well known that all these spaces coincide whenever  $G$  is amenable.) Our aim in this section is to introduce some similar spaces for  $\Sigma$ .

In [5] we introduced the notion of so-called rf-multipliers of  $\Sigma$ , giving rise to certain bounded maps from the reduced to the full crossed product. A related concept is as follows.

Let  $T : G \times A \rightarrow A$  be a map which is linear in the second variable. For each  $g \in G$ , we let  $T_g : A \rightarrow A$  be the linear map obtained by setting

$$T_g(a) = T(g, a), \quad \text{for all } a \in A.$$

Moreover, for each  $f \in C_c(\Sigma)$ , we define  $T \cdot f \in C_c(\Sigma)$  by

$$(T \cdot f)(g) = T_g(f(g)), \quad \text{for all } g \in G.$$

Then we say that  $T$  is a (reduced) *multiplier* of  $\Sigma$  whenever there exists a (necessarily unique) bounded linear map  $M_T : (C_r^*(\Sigma), \|\cdot\|) \rightarrow (C_r^*(\Sigma), \|\cdot\|)$  satisfying

$$M_T \Lambda_\Sigma(f) = \Lambda_\Sigma(T \cdot f),$$

that is,

$$M_T \left( \sum_{g \in G} f(g) \lambda_\Sigma(g) \right) = \sum_{g \in G} T_g(f(g)) \lambda_\Sigma(g),$$

for all  $f \in C_c(\Sigma)$ . Note that  $M_T$  is then uniquely determined by

$$M_T(a \lambda_\Sigma(g)) = T_g(a) \lambda_\Sigma(g), \quad \text{for all } a \in A \text{ and } g \in G.$$

Hence we have  $\|T_g(a)\| \leq \|M_T\| \|a\|$  for every  $g \in G$  and  $a \in A$ , and it follows that each  $T_g$  is bounded with  $\|T_g\| \leq \|M_T\|$ . In particular, the family  $\{T_g\}_{g \in G}$  is necessarily (uniformly) bounded.

We also note that for each  $x \in C_r^*(\Sigma)$  we have

$$\widehat{M_T(x)}(g) = T_g(\widehat{x}(g)), \quad \text{for all } g \in G. \quad (11)$$

Indeed, this is easily seen to be true when  $x \in \Lambda_\Sigma(C_c(\Sigma))$ , and the assertion then follows from a density argument. Conversely, if there exists a bounded linear map  $M_T$  from  $C_r^*(\Sigma)$  into itself such that (11) holds for every  $x$  in  $C_r^*(\Sigma)$ , then it follows readily that  $T$  is a multiplier of  $\Sigma$ . Thus this might be taken as an alternative definition.

We will let  $MA(\Sigma)$  denote the set of all (reduced) multipliers on  $\Sigma$ . This set, that always contains the trivial multiplier  $I_\Sigma$  (defined by  $I_\Sigma(g, a) = a$  for all  $g \in G, a \in A$ ), has an obvious vector space structure, and can be equipped with the norm given by

$$\|T\| = \|M_T\|.$$

We will also consider its subspace  $M_0A(\Sigma)$  consisting of (reduced) *cb-multipliers*, that is, multipliers on  $\Sigma$  satisfying that  $M_T$  is completely bounded (see [40, Chap. 8] or [41, Chap. 3]), i.e.,  $\|M_T\|_{cb} < \infty$ .

The simplest conceivable kind of multipliers on  $\Sigma$  are those arising from scalar-valued functions on the group. Consider  $\varphi : G \rightarrow \mathbb{C}$  and let  $T^\varphi : G \times A \rightarrow A$  be defined by

$$T^\varphi(g, a) = \varphi(g)a, \quad \text{for all } a \in A \text{ and } g \in G.$$

It is not difficult to see that if  $T^\varphi \in MA(\Sigma)$  (resp.  $T^\varphi \in M_0A(\Sigma)$ ), then  $\varphi \in MA(G)$  (resp.  $\varphi \in M_0A(G)$ ). It is not clear to us that  $T^\varphi \in MA(\Sigma)$  whenever  $\varphi \in MA(G)$ . However, as we will soon deduce from a more general result, we do have  $T^\varphi \in M_0A(\Sigma)$  whenever  $\varphi \in M_0A(G)$ . As a warm-up, we will first show that  $T^\varphi \in M_0A(\Sigma)$  whenever  $\varphi \in UB(G)$ .

The following lemma, in the vein of Fell's classical absorption principle, is similar to [12, Lemma 2.1].

**Lemma 4.1** *Let  $v$  be a uniformly bounded representation of  $G$  in the group of bounded invertible linear operators on some Hilbert space  $\mathcal{K}$ .*

*Then there exists an invertible adjointable operator  $V$  on the Hilbert  $A$ -module  $H^\Sigma \otimes \mathcal{K}$  satisfying*

$$V(a\lambda_\Sigma(g) \otimes I_{\mathcal{K}})V^{-1} = a\lambda_\Sigma(g) \otimes v(g), \quad \text{for all } a \in A \text{ and } g \in G. \quad (12)$$

*Proof* Let  $W$  be the invertible adjointable operator on  $H_A^G \otimes \mathcal{K} \cong H_{A \otimes \mathcal{K}}^G$  given by

$$(W\zeta)(g) = (\text{id} \otimes v(g))\zeta(g), \quad \text{for all } \zeta \in H_{A \otimes \mathcal{K}}^G \text{ and } g \in G.$$

Then, for every vector of the form  $b \otimes \delta_h \otimes \eta \in H_A^G \otimes \mathcal{K} \cong A \otimes \ell^2(G) \otimes \mathcal{K}$ , we have

$$\begin{aligned} W(\tilde{\ell}(a)\tilde{\lambda}_\ell(g) \otimes I_{\mathcal{K}})(b \otimes \delta_h \otimes \eta) &= W(\alpha_{gh}^{-1}(a\sigma(g, h))b \otimes \delta_{gh} \otimes \eta) \\ &= \alpha_{gh}^{-1}(a\sigma(g, h))b \otimes \delta_{gh} \otimes v(gh)\eta \\ &= \alpha_{gh}^{-1}(a\sigma(g, h))b \otimes \delta_{gh} \otimes v(g)v(h)\eta \end{aligned}$$



$$\begin{aligned}
&= (\tilde{\ell}(a)\tilde{\lambda}_\ell(g) \otimes v(g))(b \otimes \delta_h \otimes v(h)\eta) \\
&= (\tilde{\ell}(a)\tilde{\lambda}_\ell(g) \otimes v(g))W(b \otimes \delta_h \otimes \eta),
\end{aligned}$$

By a density argument, we get

$$W(\tilde{\ell}(a)\tilde{\lambda}_\ell(g) \otimes I_{\mathcal{K}})W^{-1} = \tilde{\ell}(a)\tilde{\lambda}_\ell(g) \otimes v(g), \quad \text{for all } a \in A \text{ and } g \in G.$$

We can now define  $V$  on  $H^\Sigma \otimes \mathcal{K}$  by  $V = (J \otimes I_{\mathcal{K}})W(J^* \otimes I_{\mathcal{K}})$  and the identity (12) readily follows.  $\square$

Note that when  $v$  is a unitary representation of  $G$  on  $\mathcal{K}$ , the operator  $V$  in Lemma 4.1 is unitary. In particular, choosing  $v = \lambda$  (the left regular representation of  $G$  on  $\ell^2(G)$ ), we get an injective homomorphism  $\delta_\Sigma : C_r^*(\Sigma) \rightarrow C_r^*(\Sigma) \otimes C_r^*(G)$ , called the (*reduced*) *dual coaction* of  $G$  on  $C_r^*(\Sigma)$ , by setting

$$\delta_\Sigma(x) = V(x \otimes I)V^*, \quad \text{for all } x \in C_r^*(\Sigma).$$

The reader may for instance consult the appendix of [17] for a survey of the vast area of coactions on  $C^*$ -algebras and their crossed products. We won't need this theory in this paper, but we will make a couple of remarks involving  $\delta_\Sigma$  in Sect. 6.

The next proposition generalizes [12, Theorem 2.2].

**Proposition 4.2** *Let  $v$  be a uniformly bounded representation of  $G$  into the bounded invertible operators on some Hilbert space  $\mathcal{K}$  and set  $K = \sup \{\|v(g)\|; g \in G\} < \infty$ .*

*Let  $\eta_1, \eta_2 \in \mathcal{K}$  and define  $\varphi \in UB(G)$  by*

$$\varphi(g) = \langle \eta_1, v(g)\eta_2 \rangle, \quad \text{for all } g \in G.$$

*Then  $T^\varphi \in M_0A(\Sigma)$ . Moreover, setting  $M_\varphi = M_{T^\varphi} : C_r^*(\Sigma) \rightarrow C_r^*(\Sigma)$ , we have*

$$M_\varphi(a\lambda_\Sigma(g)) = \varphi(g)a\lambda_\Sigma(g), \quad \text{for all } a \in A \text{ and } g \in G,$$

*and*

$$\|T^\varphi\| = \|M_\varphi\| \leq \|M_\varphi\|_{\text{cb}} \leq K^2 \|\eta_1\| \|\eta_2\|.$$

*Proof* Let  $V$  be the operator on  $H^\Sigma \otimes \mathcal{K}$  obtained in Lemma 4.1.

For  $\eta \in \mathcal{K}$ , let  $\theta_\eta : H^\Sigma \rightarrow H^\Sigma \otimes \mathcal{K}$  be the adjointable operator given by

$$\theta_\eta(\xi) = \xi \otimes \eta, \quad \text{for all } \xi \in H^\Sigma,$$

its adjoint being determined by  $\theta_\eta^*(\xi' \otimes \eta') = \langle \eta, \eta' \rangle \xi'$ , for  $\xi' \in H^\Sigma$ ,  $\eta' \in \mathcal{K}$ .

Now define  $M_\varphi : \mathcal{L}(H^\Sigma) \rightarrow \mathcal{L}(H^\Sigma)$  by

$$M_\varphi(x) = \theta_{\eta_1}^* V(x \otimes I_{\mathcal{K}}) V^{-1} \theta_{\eta_2}.$$

Then  $M_\varphi$  is completely bounded (see [40, Chap. 8] or [41, Chap. 3]), with

$$\|M_\varphi\|_{\text{cb}} \leq \|\theta_{\eta_1}^* V\| \|V^{-1} \theta_{\eta_2}\| \leq K^2 \|\eta_1\| \|\eta_2\|.$$

Let  $\xi \in H^\Sigma$ . Using Lemma 4.1, we get

$$\begin{aligned} M_\varphi(a\lambda_\Sigma(g))\xi &= \theta_{\eta_1}^* V(a\lambda_\Sigma(g) \otimes I_{\mathcal{K}}) V^{-1} \theta_{\eta_2} \xi \\ &= \theta_{\eta_1}^* (a\lambda_\Sigma(g) \otimes v(g)) \theta_{\eta_2} \xi \\ &= \theta_{\eta_1}^* (a\lambda_\Sigma(g) \xi \otimes v(g) \eta_2) \\ &= \langle \eta_1, v(g) \eta_2 \rangle a\lambda_\Sigma(g) \xi \\ &= \varphi(g) a\lambda_\Sigma(g) \xi \end{aligned}$$

This shows that  $T^\varphi \in M_0A(\Sigma)$ , with  $M_{T^\varphi} = M_\varphi$ , and the final assertion follows from this.  $\square$

When  $\sigma$  is trivial, the following corollary is due to Haagerup (see [25, Lemma 3.5]).

**Corollary 4.3** *Let  $\varphi$  be a positive definite function on  $G$ .*

*Then  $T^\varphi \in M_0A(\Sigma)$ ,  $M_{T^\varphi} : C_r^*(\Sigma) \rightarrow C_r^*(\Sigma)$  is completely positive, and*

$$\|T^\varphi\| = \|M_{T^\varphi}\| = \|M_{T^\varphi}\|_{\text{cb}} = \varphi(e).$$

*Proof* As is well known, there exists a unitary representation  $v$  of  $G$  on some Hilbert space  $\mathcal{H}$  such that  $\varphi(g) = \langle \eta, v(g)\eta \rangle$  for all  $g \in G$ . Hence, the result follows readily from Proposition 4.2 and its proof.  $\square$

More generally, given a map  $\varphi : G \rightarrow A$ , we may consider the maps  $L^\varphi$  and  $R^\varphi$  from  $G \times A$  into  $A$  given by

$$L^\varphi(g, a) = \varphi(g)a, \quad R^\varphi(g, a) = a\varphi(g), \quad \text{for all } g \in G \text{ and } a \in A.$$

Inspired by the known characterizations of  $M_0A(G)$  (see e.g. [7, 8, 33, 41]), we will give in Theorem 4.5 some conditions ensuring that these maps belong to  $M_0A(\Sigma)$ . For a different result about  $MA(\Sigma)$ , see Proposition 4.16.

We will use the following lemma:

**Lemma 4.4** *Let  $X$  be a Hilbert  $A$ -module and let  $\zeta \in \ell^\infty(G, X)$ . Define  $V_\zeta : H_A^G \rightarrow H_X^G$  by*

$$[V_\zeta \xi](g) = \zeta(g) \cdot \xi(g), \quad \text{for all } \xi \in H_A^G \text{ and } g \in G.$$

*Then  $V_\zeta \in \mathcal{L}(H_A^G, H_X^G)$ ,  $\|V_\zeta\| \leq \|\zeta\|_\infty$  and  $V_\zeta^*$  is given by  $V_\zeta^* = W_\zeta : H_X^G \rightarrow H_A^G$ , where*

$$[W_\zeta \eta](g) = \langle \zeta(g), \eta(g) \rangle, \quad \text{for all } \eta \in H_X^G \text{ and } g \in G.$$

*Proof* Let  $\xi \in H_A^G$  and define  $\gamma : G \rightarrow X$  by

$$\gamma(g) = \zeta(g) \cdot \xi(g), \quad \text{for all } g \in G.$$

Then, for each  $g \in G$ , we have

$$\langle \gamma(g), \gamma(g) \rangle = \xi(g)^* \langle \zeta(g), \zeta(g) \rangle \xi(g) \leq \|\zeta(g)\|^2 \xi(g)^* \xi(g) \leq \|\zeta\|_\infty^2 \xi(g)^* \xi(g).$$

It follows that  $\gamma \in H_X^G$  and

$$\|\gamma\|^2 = \left\| \sum_{g \in G} \langle \gamma(g), \gamma(g) \rangle \right\| \leq \|\zeta\|_\infty^2 \left\| \sum_{g \in G} \xi(g)^* \xi(g) \right\|.$$

This shows that  $V_\zeta$  is well-defined and  $\|V_\zeta \xi\| \leq \|\zeta\|_\infty \|\xi\|$ .

Similarly, let  $\eta \in H_X^G$  and define  $\delta : G \rightarrow A$  by

$$\delta(g) = \langle \zeta(g), \eta(g) \rangle, \quad \text{for all } g \in G.$$

Then, for each  $g \in G$ , we have

$$\delta(g)^* \delta(g) = \langle \zeta(g), \eta(g) \rangle^* \langle \zeta(g), \eta(g) \rangle \leq \|\zeta(g)\|^2 \langle \eta(g), \eta(g) \rangle \leq \|\zeta\|_\infty^2 \langle \eta(g), \eta(g) \rangle.$$

It follows that  $\delta \in H_A^G$ . This shows that  $W_\zeta : H_X^G \rightarrow H_A^G$  is well-defined.

Now, let  $\xi \in H_A^G$  and  $\eta \in H_X^G$ . Then we have

$$\begin{aligned} \langle V_\zeta \xi, \eta \rangle &= \sum_{g \in G} \langle (V_\zeta \xi)(g), \eta(g) \rangle = \sum_{g \in G} \langle \zeta(g) \cdot \xi(g), \eta(g) \rangle \\ &= \sum_{g \in G} \xi(g)^* \langle \zeta(g), \eta(g) \rangle = \sum_{g \in G} \xi(g)^* (W_\zeta \eta)(g) = \langle \xi, W_\zeta \eta \rangle. \end{aligned}$$

Hence,  $V_\zeta$  is adjointable with  $V_\zeta^* = W_\zeta$ , as desired.  $\square$

**Theorem 4.5** Let  $\varphi : G \rightarrow A$  be a map and let  $L^\varphi, R^\varphi : G \times A \rightarrow A$  be defined as above. Let  $\pi$  be a representation of  $A$  on some Hilbert  $A$ -module  $X$  and let  $\eta_1, \eta_2 \in \ell^\infty(G, X)$ .

(I) Assume that

$$\pi(a)\eta_2(t) = \eta_2(t) \cdot a, \quad \text{for all } a \in A \text{ and } t \in G, \quad (13)$$

and

$$\varphi(st^{-1}) = \alpha_s(\langle \eta_1(s), \eta_2(t) \rangle), \quad \text{for all } s, t \in G. \quad (14)$$

Then  $L^\varphi \in M_0 A(\Sigma)$ , and  $\|L^\varphi\| \leq \|M_{L^\varphi}\|_{\text{cb}} \leq \|\eta_1\|_\infty \|\eta_2\|_\infty$ .

If  $\eta_1 = \eta_2$ , then  $M_{L^\varphi}$  is completely positive and  $\|L^\varphi\| = \|M_{L^\varphi}\|_{\text{cb}} = \|\varphi(e)\|$ .

(r) Assume that

$$\pi(a)\eta_1(t) = \eta_1(t) \cdot a, \quad \text{for all } a \in A \text{ and } t \in G, \quad (15)$$

and

$$\varphi(st^{-1}) = \text{Ad}(\sigma(s, st^{-1}))\alpha_s((\eta_1(s), \eta_2(t))), \quad \text{for all } s, t \in G. \quad (16)$$

Then  $R^\varphi \in M_0A(\Sigma)$ , and  $\|R^\varphi\| \leq \|M_{R^\varphi}\|_{\text{cb}} \leq \|\eta_1\|_\infty \|\eta_2\|_\infty$ .

If  $\eta_1 = \eta_2$ , then  $M_{R^\varphi}$  is completely positive and  $\|R^\varphi\| = \|M_{R^\varphi}\|_{\text{cb}} = \|\varphi(e)\|$ .

*Proof* We use the notation of Lemma 4.4 and set  $V_j = V_{\eta_j} \in \mathcal{L}(H_A^G, H_X^G)$ ,  $j = 1, 2$ .

We may then define a completely bounded linear map  $M : \mathcal{L}(H_X^G) \rightarrow \mathcal{L}(H_A^G)$  by

$$M(S) = V_1^* S V_2, \quad S \in \mathcal{L}(H_X^G),$$

and Lemma 4.4 gives  $\|M\|_{\text{cb}} \leq \|V_1\| \|V_2\| \leq \|\eta_1\|_\infty \|\eta_2\|_\infty$ . Clearly,  $M$  is completely positive if  $\eta_1 = \eta_2$ .

Consider  $a \in A$ ,  $g \in G$  and  $\xi \in H_A^G$ . Then, for each  $h \in G$ , we have

$$[M(\tilde{\pi}(a)\tilde{\lambda}_\pi(g))\xi](h) = [V_1^* \tilde{\pi}(a)\tilde{\lambda}_\pi(g) V_2 \xi](h) = \langle \eta_1(h), (\tilde{\pi}(a)\tilde{\lambda}_\pi(g) V_2 \xi)(h) \rangle.$$

We also note that

$$\begin{aligned} (\tilde{\pi}(a)\tilde{\lambda}_\pi(g) V_2 \xi)(h) &= \pi(\alpha_h^{-1}(a)\alpha_h^{-1}(\sigma(g, g^{-1}h)))(V_2 \xi)(g^{-1}h) \\ &= \pi(\alpha_h^{-1}(a\sigma(g, g^{-1}h)))(\eta_2(g^{-1}h) \cdot \xi(g^{-1}h)) \\ &= (\pi(\alpha_h^{-1}(a\sigma(g, g^{-1}h)))\eta_2(g^{-1}h)) \cdot \xi(g^{-1}h) \end{aligned}$$

where we have used  $A$ -linearity at the last step. Hence, we get

$$[M(\tilde{\pi}(a)\tilde{\lambda}_\pi(g))\xi](h) = \langle \eta_1(h), \pi(\alpha_h^{-1}(a\sigma(g, g^{-1}h)))\eta_2(g^{-1}h) \rangle \xi(g^{-1}h). \quad (17)$$

We divide the rest of the proof in two cases.

(i) Assume that the assumptions in (I) are satisfied. Assumption (13) gives

$$\pi(\alpha_h^{-1}(a\sigma(g, g^{-1}h)))\eta_2(g^{-1}h) = \eta_2(g^{-1}h) \cdot \alpha_h^{-1}(a\sigma(g, g^{-1}h)),$$

so we get

$$\begin{aligned} [M(\tilde{\pi}(a)\tilde{\lambda}_\pi(g))\xi](h) &= \langle \eta_1(h), \eta_2(g^{-1}h) \cdot \alpha_h^{-1}(a\sigma(g, g^{-1}h)) \rangle \xi(g^{-1}h) \\ &= \langle \eta_1(h), \eta_2(g^{-1}h) \rangle \alpha_h^{-1}(a\sigma(g, g^{-1}h)) \xi(g^{-1}h). \end{aligned}$$

As assumption (14) gives  $\varphi(g) = \varphi(h(g^{-1}h)^{-1}) = \alpha_h(\langle \eta_1(h), \eta_2(g^{-1}h) \rangle)$ , we have

$$\alpha_h^{-1}(\varphi(g)) = \langle \eta_1(h), \eta_2(g^{-1}h) \rangle.$$

Hence, we get

$$\begin{aligned} [M(\tilde{\pi}(a)\tilde{\lambda}_{\pi}(g))\xi](h) &= \alpha_h^{-1}(\varphi(g))\alpha_h^{-1}(a\sigma(g, g^{-1}h))\xi(g^{-1}h) \\ &= \alpha_h^{-1}(\varphi(g)a\sigma(g, g^{-1}h))\xi(g^{-1}h) \\ &= [\tilde{\ell}(\varphi(g)a)\tilde{\lambda}_{\ell}(g)\xi](h). \end{aligned}$$

By linearity, we get  $M(\tilde{\pi} \times \tilde{\lambda}_{\pi})(f) = \Lambda(L^{\varphi} \cdot f)$  for all  $f \in C_c(\Sigma)$ . As  $M$  is bounded, it follows that  $M$  maps  $(\tilde{\pi} \times \tilde{\lambda}_{\pi})(C^*(\Sigma))$  into  $\Lambda(C^*(\Sigma)) \subset \mathcal{L}(H_A^G)$ .

As  $\tilde{\pi} \times \tilde{\lambda}_{\pi}$  is weakly contained in  $\Lambda$  (cf. [5, p. 188]), and  $\Lambda_{\Sigma}$  is unitarily equivalent to  $\Lambda$  (via  $J$ ), there exists a homomorphism  $\psi : C_r^*(\Sigma) \rightarrow \mathcal{L}(H_X^G)$  such that  $\psi \Lambda_{\Sigma} = \tilde{\pi} \times \tilde{\lambda}_{\pi}$ .

Let now  $\tilde{M} : C_r^*(\Sigma) \rightarrow C_r^*(\Sigma)$  be given by  $\tilde{M} = (\text{Ad}J)M\psi$ . Since  $M$  is completely bounded,  $\tilde{M}$  is also completely bounded. Moreover, we have

$$\begin{aligned} \tilde{M} \Lambda_{\Sigma}(f) &= (\text{Ad}J)M\psi \Lambda_{\Sigma}(f) = (\text{Ad}J)M(\tilde{\pi} \times \tilde{\lambda}_{\pi})(f) \\ &= (\text{Ad}J) \Lambda(L^{\varphi} \cdot f) = \Lambda_{\Sigma}(L^{\varphi} \cdot f) \end{aligned}$$

for all  $f \in C_c(\Sigma)$ . This shows that  $L^{\varphi} \in M_0A(\Sigma)$  with  $M_{L^{\varphi}} = \tilde{M}$ . Moreover, we get

$$|||L^{\varphi}||| = \|M_{L^{\varphi}}\| \leq \|\tilde{M}\|_{\text{cb}} \leq \|M\|_{\text{cb}} \leq \|\eta_1\|_{\infty} \|\eta_2\|_{\infty}.$$

If  $\eta_1 = \eta_2$ , then  $M_{L^{\varphi}} = \tilde{M}$  is a composition of completely positive maps and therefore itself completely positive. As  $M_{L^{\varphi}}(1) = \varphi(e)$ , the last assertion of (I) readily follows.

(ii) Assume now that the assumptions in  $r$ ) are satisfied. Equation (17) gives

$$[M(\tilde{\pi}(a)\tilde{\lambda}_{\pi}(g))\xi](h) = \langle \pi(\alpha_h^{-1}(a\sigma(g, g^{-1}h)))^* \eta_1(h), \eta_2(g^{-1}h) \rangle \xi(g^{-1}h).$$

Using assumption (15), it follows that

$$\begin{aligned} [M(\tilde{\pi}(a)\tilde{\lambda}_{\pi}(g))\xi](h) &= \langle \eta_1(h) \cdot \alpha_h^{-1}(a\sigma(g, g^{-1}h))^*, \eta_2(g^{-1}h) \rangle \xi(g^{-1}h) \\ &= \alpha_h^{-1}(a\sigma(g, g^{-1}h)) \langle \eta_1(h), \eta_2(g^{-1}h) \rangle \xi(g^{-1}h). \end{aligned}$$

Now, assumption (16) gives

$$\varphi(g) = \varphi(h(gh^{-1})^{-1}) = \sigma(g, g^{-1}h)\alpha_h(\langle \eta_1(h), \eta_2(g^{-1}h) \rangle)\sigma(g, g^{-1}h)^*,$$

hence

$$\langle \eta_1(h), \eta_2(g^{-1}h) \rangle = \alpha_h^{-1}(\sigma(g, g^{-1}h)^* \varphi(g) \sigma(g, g^{-1}h)),$$

and we obtain

$$\begin{aligned} & [M(\tilde{\pi}(a)\tilde{\lambda}_{\pi}(g))\xi](h) \\ &= \alpha_h^{-1}(a\sigma(g, g^{-1}h))\alpha_h^{-1}(\sigma(g, g^{-1}h)^*\varphi(g)\sigma(g, g^{-1}h))\xi(g^{-1}h) \\ &= \alpha_h^{-1}(a\varphi(g))\alpha_h^{-1}(\sigma(g, g^{-1}h))\xi(g^{-1}h) \\ &= [\tilde{\ell}(a\varphi(g))\tilde{\lambda}_{\ell}(g)\xi](h). \end{aligned}$$

By linearity, we get  $M(\tilde{\pi} \times \tilde{\lambda}_{\pi})(f) = \Lambda(R^{\varphi} \cdot f)$  for all  $f \in C_c(\Sigma)$ . Clearly, we can now proceed as in the previous case to finish the proof.  $\square$

**Example 4.6** Let  $(\rho, v)$  be an equivariant representation of  $\Sigma$  on a Hilbert module  $X$  and let  $x, y \in X$ . Assume that  $x$  or  $y$  lies in the central part  $Z_X$  of  $X$ . Let  $\varphi : G \rightarrow A$  be given by

$$\varphi(g) = \langle x, v(g)y \rangle, \quad \text{for all } g \in G,$$

and consider the associated map  $L^{\varphi}$  if  $y \in Z_X$  (resp.  $R^{\varphi}$  if  $x \in Z_X$ ). Then an elementary computation, using that  $Z_X$  is left invariant by each  $v(g)$  (see [5]), gives that  $\varphi$  satisfies the assumptions in part l) (resp. part (r)) of Theorem 4.5 with

$$\xi_1(s) = \text{ad}_{\rho}(\sigma(s, s^{-1}))v(s^{-1})x, \quad \xi_2(t) = v(t^{-1})y, \quad \text{when } y \in Z_X$$

(resp.

$$\xi_1(s) = v(s^{-1})x, \quad \xi_2(t) = \text{ad}_{\rho}(\sigma(t^{-1}, t)^*)v(s^{-1})x, \quad \text{when } x \in Z_X).$$

Hence, we get that  $L^{\varphi}$  (resp.  $R^{\varphi}$ ) is a cb-multiplier of  $\Sigma$  when  $y \in Z_X$  (resp.  $x \in Z_X$ ). We will in fact give an alternative approach in Example 4.11.

Theorem 4.5 is well known when  $A = \mathbb{C}$  and  $\sigma = 1$ . Indeed, consider  $\varphi : G \rightarrow \mathbb{C}$ . As shown in [41, Theorems 5.1 and 6.4] (see also [7, 33]),  $\varphi \in M_0A(G)$  if and only if there exist a Hilbert space  $\mathcal{K}$  and  $\xi_1, \xi_2 \in \ell^{\infty}(G, \mathcal{K})$  such that

$$\varphi(st^{-1}) = \langle \xi_1(s), \xi_2(t) \rangle \quad \text{for all } s, t \in G. \quad (18)$$

Moreover, in this case, letting  $M_{\varphi} : C_r^*(G) \rightarrow C_r^*(G)$  denote the associated completely bounded map, we have  $\|M_{\varphi}\|_{\text{cb}} = \inf \|\xi_1\|_{\infty} \|\xi_2\|_{\infty}$ , where the infimum is taken over all possible pairs  $\xi_1, \xi_2$  satisfying the above conditions for some Hilbert space  $\mathcal{K}$ .

Using this characterization of  $M_0A(G)$ , we get:

**Corollary 4.7** *Let  $\varphi \in M_0A(G)$ . Then  $T^{\varphi} \in M_0A(\Sigma)$  and  $\|T^{\varphi}\| \leq \|M_{T^{\varphi}}\|_{\text{cb}} \leq \|M_{\varphi}\|_{\text{cb}}$ .*

*Proof* Define  $\varphi_A : G \rightarrow A$  by  $\varphi_A(s) = \varphi(s) \cdot 1$ . Then pick a Hilbert space  $\mathcal{K}$  and  $\xi_1, \xi_2 \in \ell^{\infty}(G, \mathcal{K})$  such that (18) holds.

Consider the Hilbert  $A$ -module  $X = A \otimes \mathcal{K}$  and the canonical representation  $\pi$  of  $A$  on  $X$  (determined by  $\pi(a)(b \otimes \zeta) = ab \otimes \zeta$ ). Moreover, define  $\eta_1, \eta_2 \in \ell^\infty(G, X)$  by  $\eta_j(s) = 1 \otimes \xi_j(s)$ ,  $j = 1, 2$ .

Then, trivially,  $\eta_2$  satisfies the assumption (13) in part (I) of Theorem 4.5. Further, for all  $s, t \in G$ , we have

$$\alpha_s(\langle (\eta_1(s), \eta_2(t)) \rangle) = \alpha_s(\langle (1, 1) \langle \xi_1(s), \xi_2(t) \rangle \rangle) = \langle \xi_1(s), \xi_2(t) \rangle \cdot 1 = \varphi_A(st^{-1}).$$

Hence, the assumption (14) in  $I$  is also satisfied (with  $\varphi_A$ ) and we may apply Theorem 4.5. We get  $T^\varphi = L^{\varphi_A} \in M_0A(\Sigma)$ , and

$$\|T^\varphi\| \leq \|M_{T^\varphi}\|_{\text{cb}} = \|M_{L^{\varphi_A}}\|_{\text{cb}} \leq \|\eta_1\|_\infty \|\eta_2\|_\infty = \|\xi_1\|_\infty \|\xi_2\|_\infty.$$

As this holds for any choice of  $\xi_1, \xi_2$  satisfying (18), we get  $\|T^\varphi\| \leq \|M_{T^\varphi}\|_{\text{cb}} \leq \|M_\varphi\|_{\text{cb}}$ , as asserted.  $\square$

We will now show how one may produce cb-multipliers of  $\Sigma$  associated with equivariant representations of  $\Sigma$ , in a more general way than the one outlined in Example 4.6. When  $A$  is trivial, the basic ingredient in an equivariant representation consists of a unitary representation of the group on some Hilbert space, and the associated multipliers are then just given by Proposition 4.2. In the general case, our procedure is technically much more involved and requires some preparations. We first state the result.

**Theorem 4.8** *Let  $(\rho, v)$  be an equivariant representation of  $\Sigma$  on a Hilbert  $A$ -module  $X$  and let  $x, y \in X$ . Define  $T : G \times A \rightarrow A$  by*

$$T(g, a) = \langle x, \rho(a)v(g)y \rangle, \quad \text{for all } g \in G \text{ and } a \in A.$$

*Then  $T \in M_0A(\Sigma)$  and*

$$\|T\| = \|M_T\| \leq \|M_T\|_{\text{cb}} \leq \|x\| \|y\|.$$

*Hence,  $M_T : C_r^*(\Sigma) \rightarrow C_r^*(\Sigma)$  satisfies*

$$M_T(a\lambda_\Sigma(g)) = \langle x, \rho(a)v(g)y \rangle \lambda_\Sigma(g) \quad (19)$$

*for each  $a \in A$  and  $g \in G$ .*

*If  $x = y$ , then  $M_T$  is completely positive and  $\|T\| = \|M_T\| = \|M_T\|_{\text{cb}} = \|x\|^2$ .*

We define  $B(\Sigma)$  to be the set of all multipliers of  $\Sigma$  obtained as in Theorem 4.8, thinking of it as the set of  $A$ -valued matrix coefficients associated with equivariant representations of  $\Sigma$ . Theorem 4.8 then says that

$$B(\Sigma) \subset M_0A(\Sigma).$$

Note that  $I_\Sigma \in B(\Sigma)$  (taking  $X = A$ ,  $(\rho, v) = (\ell, \alpha)$ ,  $\xi = \eta = 1$ ). Moreover,  $M_0A(\Sigma)$  (and  $MA(\Sigma)$ ) can be endowed with an algebra structure,  $B(\Sigma)$  and  $UB(G)$  may be seen as unital subalgebras of  $M_0A(\Sigma)$ , while  $B(G)$  may be seen as a unital subalgebra of both  $B(\Sigma)$  and  $UB(G)$ . To do this, we would need to discuss the notion of tensor product of equivariant representations of  $\Sigma$ . As this is somewhat lengthy and would take us away from our main focus in this paper, we will not elaborate on this any further here.

Our proof of Theorem 4.8 will rely on a new version of the Fell's absorption principle. It uses the machinery developed in [5, Sect. 4], where another analogue of Fell's principle was established [5, Theorem 4.11].

We start with a lemma. Note that if  $X$  is a Hilbert  $A$ -module,  $\ell_\Sigma$  being a representation of  $A$  on the Hilbert  $A$ -module  $H^\Sigma$ , we may form the internal tensor product  $X \otimes_{\ell_\Sigma} H^\Sigma$ , which is also a Hilbert  $A$ -module (see [35, Chap. 4]).

**Lemma 4.9** *Let  $(\rho, v)$  be an equivariant representation of  $\Sigma$  on a Hilbert  $A$ -module  $X$ .*

*There exists a unitary operator  $W \in \mathcal{L}(X \otimes_{\ell_\Sigma} H^\Sigma, H_X^G)$  which satisfies*

$$[W(x \dot{\otimes} \xi)](g) = v(g)^{-1}(x \cdot \xi(g)) \quad (20)$$

for all  $x \in X$ ,  $\xi \in H^\Sigma$  and  $g \in G$ .

*Proof* We first define  $W$  on the dense subspace  $Y$  of  $X \otimes_{\ell_\Sigma} H^\Sigma$  consisting of the span of elements of the form  $x \dot{\otimes} \xi$ , where  $x \in X$ ,  $\xi \in C_c(\Sigma)$ . For  $y = \sum_{i=1}^n x_i \dot{\otimes} \xi_i \in Y$ , we set

$$(Wy)(g) = v(g)^{-1} \left( \sum_{i=1}^n x_i \cdot \xi_i(g) \right), \quad \text{for all } g \in G.$$

Then we have

$$\begin{aligned} \langle Wy, Wy \rangle &= \sum_{g \in G} \sum_{i,j=1}^n \left\langle v(g)^{-1}(x_i \cdot \xi_i(g)), v(g)^{-1}(x_j \cdot \xi_j(g)) \right\rangle \\ &= \sum_{g \in G} \sum_{i,j=1}^n \alpha_g^{-1}(\langle x_i \cdot \xi_i(g), x_j \cdot \xi_j(g) \rangle) \\ &= \sum_{g \in G} \sum_{i,j=1}^n \alpha_g^{-1}(\xi_i(g)^* \langle x_i, x_j \rangle \xi_j(g)) \\ &= \sum_{g \in G} \sum_{i,j=1}^n \alpha_g^{-1}(\xi_i(g)^* (\ell_\Sigma(\langle x_i, x_j \rangle \xi_j)(g))) \\ &= \sum_{i,j=1}^n \sum_{g \in G} \alpha_g^{-1}(\xi_i(g)^* (\ell_\Sigma(\langle x_i, x_j \rangle \xi_j)(g))) \end{aligned}$$



$$= \sum_{i,j=1}^n \langle \xi_i, \ell_\Sigma(\langle x_i, x_j \rangle) \xi_j \rangle = \sum_{i,j=1}^n \langle x_i \dot{\otimes} \xi_i, x_j \dot{\otimes} \xi_j \rangle = \langle y, y \rangle$$

as interchanging the sums is allowed, the  $\xi_i$ 's being assumed to have finite support.

It follows that  $W$  is a well defined isometry from  $Y$  into  $H_X^G$ , that satisfies equation (20) by definition. It extends to an isometry, also denoted by  $W$ , from  $X \otimes_{\ell_\Sigma} H^\Sigma$  into  $H_X^G$ . As the range of  $W$  obviously contains  $C_c(G, X)$ ,  $W$  is surjective.

Moreover,  $W$  is  $A$ -linear: It clearly suffices to check that  $W((x \dot{\otimes} \xi) \cdot a) = (W(x \dot{\otimes} \xi)) \cdot a$  for all  $x \in X$ ,  $\xi \in H^\Sigma$  and  $a \in A$ . Now, for every  $g \in G$ , we have

$$\begin{aligned} [W((x \dot{\otimes} \xi) \cdot a)](g) &= [W((x \dot{\otimes} (\xi \times a)))](g) = v(g)^{-1}(x \cdot (\xi(g) \alpha_g(a))) \\ &= v(g)^{-1}((x \cdot \xi(g)) \cdot \alpha_g(a)) = (v(g)^{-1}(x \cdot \xi(g))) \cdot a \\ &= ([W(x \dot{\otimes} \xi)](g)) \cdot a = [(W(x \dot{\otimes} \xi)) \cdot a](g), \end{aligned}$$

where we have used property (iv) of equivariant representations. This shows our assertion.

Thus,  $W$  is a bijective,  $A$ -linear isometry and it follows from [35, Theorem 3.5] that  $W$  is unitary.  $\square$

Here is our new version of Fell's classical absorption principle.

**Theorem 4.10** *Let  $(\rho, v)$  be an equivariant representation of  $\Sigma$  on a Hilbert  $A$ -module  $X$  and let  $(\ell_\Sigma)_* : \mathcal{L}(X) \rightarrow \mathcal{L}(X \otimes_{\ell_\Sigma} H^\Sigma)$  denote the canonical homomorphism associated with  $\ell_\Sigma$ , so  $\rho \dot{\otimes} \ell_\Sigma = (\ell_\Sigma)_* \circ \rho : A \rightarrow \mathcal{L}(X \otimes_{\ell_\Sigma} H^\Sigma)$ .*

*Then the product covariant representation  $(\rho \dot{\otimes} \ell_\Sigma, v \dot{\otimes} \lambda_\Sigma)$  of  $\Sigma$  on  $X \otimes_{\ell_\Sigma} H^\Sigma$  is unitarily equivalent to the regular covariant representation  $(\tilde{\rho}, \tilde{\lambda}_\rho)$  of  $\Sigma$  on  $H_X^G$ . Hence, we have*

$$(\rho \dot{\otimes} \ell_\Sigma) \times (v \dot{\otimes} \lambda_\Sigma) \simeq \tilde{\rho} \times \tilde{\lambda}_\rho.$$

*Proof* Let  $W \in \mathcal{L}(X \otimes_{\ell_\Sigma} H^\Sigma, H_X^G)$  be the unitary operator defined in Lemma 4.9. To prove the assertion, it is enough to prove the equalities  $\tilde{\rho}(a)W = W(\rho \dot{\otimes} \ell_\Sigma)(a)$  and  $\tilde{\lambda}_\rho(g)W = W(v \dot{\otimes} \lambda_\Sigma)(g)$  for all  $a \in A$  and  $g \in G$ . We will check these on a total set of vectors.

We have

$$\begin{aligned} [\tilde{\rho}(a)W(x \dot{\otimes} \xi)](h) &= \rho(\alpha_h^{-1}(a))([W(x \dot{\otimes} \xi)](h)) \\ &= \rho(\alpha_h^{-1}(a))([v(g)^{-1}(x \cdot \xi(h))]) \\ &= v(h)^{-1}\rho(a)(x \cdot \xi(h)) \\ &= v(h)^{-1}((\rho(a)x) \cdot \xi(h)) \\ &= [W(\rho(a)x \dot{\otimes} \xi)](h) \end{aligned}$$

$$\begin{aligned}
&= [W((\ell_\Sigma)_* \circ \rho)(a)(x \dot{\otimes} \xi)](h) \\
&= [W(\rho \dot{\otimes} \ell_\Sigma)(a)(x \dot{\otimes} \xi)](h),
\end{aligned}$$

for all  $a \in A$ ,  $x \in X$ ,  $\xi \in H^\Sigma$  and  $h \in G$ .

Similarly, using property (ii) of equivariant representations, we have

$$v(g) = \text{ad}_\rho(\sigma(g, g^{-1}h))v(h)v(g^{-1}h)^{-1}, \quad \text{for all } g, h \in G,$$

and this gives

$$\begin{aligned}
[\tilde{\lambda}_\rho(g)W(x \dot{\otimes} \xi)](h) &= \rho(\alpha_h^{-1}(\sigma(g, g^{-1}h)))([W(x \dot{\otimes} \xi)](g^{-1}h)) \\
&= v(h)^{-1}\rho(\sigma(g, g^{-1}h))v(h)v(g^{-1}h)^{-1}(x \cdot \xi(g^{-1}h)) \\
&= v(h)^{-1}\left((v(g)(x \cdot (\xi(g^{-1}h)))) \cdot \sigma(g, g^{-1}h)\right) \\
&= v(h)^{-1}((v(g)x) \cdot (\lambda_\Sigma(g)\xi)(h)) \\
&= [W(v(g)x \dot{\otimes} \lambda_\Sigma(g)\xi)](h) \\
&= [W(v \dot{\otimes} \lambda_\Sigma)(g)(x \dot{\otimes} \xi)](h),
\end{aligned}$$

for all  $x \in X$ ,  $\xi \in H^\Sigma$  and  $g, h \in G$ .  $\square$

To prove Theorem 4.8, we will use the same notation as in the proof of Theorem 4.10.

*Proof of Theorem 4.8* As discussed in [5, Sect. 2], the Hilbert  $A$ -modules  $H_X^G$  and  $H_A^G \otimes_\rho X$  are unitarily equivalent via the map  $U \in \mathcal{L}(H_A^G \otimes_\rho X, H_X^G)$  given by

$$[U(f \dot{\otimes} x)](h) = \rho(f(h))x, \quad \text{for all } f \in H_A^G, x \in X \text{ and } h \in G.$$

Moreover, letting  $\rho_* : \mathcal{L}(H_A^G) \rightarrow \mathcal{L}(H_A^G \otimes_\rho X)$  denote the canonical homomorphism, we have

$$\begin{aligned}
U\rho_*(\tilde{\ell}(a))U^* &= \tilde{\rho}(a) & \text{for all } a \in A, \\
U\rho_*(\tilde{\lambda}_\ell(g))U^* &= \tilde{\lambda}_\rho(g) & \text{for all } g \in G.
\end{aligned}$$

For  $x \in X$ , let  $\theta_x \in \mathcal{L}(H^\Sigma, X \otimes_{\ell_\Sigma} H^\Sigma)$  be defined as in [35, Lemma 4.6], that is,

$$\theta_x(\xi) = x \dot{\otimes} \xi, \quad \text{for all } \xi \in H^\Sigma.$$

Then, for all  $y \in X$ ,  $\eta \in H^\Sigma$ , we have

$$\theta_x^*(y \dot{\otimes} \eta) = \ell_\Sigma(\langle x, y \rangle)\eta = \langle x, y \rangle\eta$$

(since we identify  $A$  with  $\ell_\Sigma(A)$ ).

Let  $x, y \in X$  be given. Then define a linear map  $\Phi : \mathcal{L}(H_A^G) \rightarrow \mathcal{L}(H^\Sigma)$  by

$$\Phi(\cdot) = \theta_x^* W^* U \rho_*(\cdot) U^* W \theta_y.$$

Then  $\Phi$  is completely bounded (see e.g. [40, Chap. 8] or [41, Chap. 3]), with

$$\|\Phi\| \leq \|\Phi\|_{\text{cb}} \leq \|x\| \|y\|.$$

Moreover, if  $x = y$ , then  $\Phi(\cdot) = \theta_x^* W^* U \rho_*(\cdot) U^* W \theta_x$  becomes completely positive and satisfies

$$\|\Phi\| = \|\Phi\|_{\text{cb}} = \|\Phi(I)\| = \|x\|^2.$$

Now, for all  $a \in A$ ,  $g \in G$  and  $\xi \in H^\Sigma$ , we compute

$$\begin{aligned} \Phi(\tilde{\ell}(a)\tilde{\lambda}_\ell(g))\xi &= \theta_x^* W^* U \rho_*(\tilde{\ell}(a)\tilde{\lambda}_\ell(g)) U^* W \theta_y \xi \\ &= \theta_x^* W^* \tilde{\rho}(a)\tilde{\lambda}_\rho(a) W(y \dot{\otimes} \xi) \\ &= \theta_x^*(\rho \dot{\otimes} \ell_\Sigma)(a)(v \dot{\otimes} \lambda_\Sigma)(g)(y \dot{\otimes} \xi) \\ &= \theta_x^*[(\ell_\Sigma)_*(\rho(a))](v(g)y \dot{\otimes} \lambda_\Sigma(g)\xi) \\ &= \theta_x^*(\rho(a)v(g)y \dot{\otimes} \lambda_\Sigma(g)\xi) \\ &= \langle x, \rho(a)v(g)y \rangle \lambda_\Sigma(g)\xi. \end{aligned}$$

Hence, letting  $M_T$  be the restriction of  $\Phi \circ \text{Ad}(J^*)$  to  $C_r^*(\Sigma) \subset \mathcal{L}(H^\Sigma)$ , we clearly get a completely bounded map  $M_T : C_r^*(\Sigma) \rightarrow C_r^*(\Sigma)$  satisfying

$$M_T(a\lambda_\Sigma(g)) = \langle x, \rho(a)v(g)y \rangle \lambda_\Sigma(g) = T_g(a)\lambda_\Sigma(g)$$

for all  $a \in A$  and  $g \in G$ . This means that  $T \in M_0A(\Sigma)$ , and  $M_T$  satisfies the desired properties.  $\square$

*Example 4.11* (Example 4.6 revisited). Let  $(\rho, v)$  be an equivariant representation of  $\Sigma$  on a Hilbert module  $X$  and let  $x, y \in X$ . Let  $T$  denote the associated multiplier of  $\Sigma$ , as in Theorem 4.8. Assume that  $x$  or  $y$  lies in the central part  $Z_X$  of  $X$  and let  $\varphi : G \rightarrow A$  be given by  $\varphi(g) = \langle x, v(g)y \rangle$ . Then, as  $Z_X$  is left invariant by each  $v(g)$  (cf. [5, Sect. 4]),  $T$  is given by

$$T(g, a) = \langle x, v(g)y \rangle = \varphi(g)a \quad \text{if } y \in Z_X,$$

or as

$$T(g, a) = a\langle x, v(g)y \rangle = a\varphi(g) \quad \text{if } x \in Z_X.$$

Hence, we recover the multipliers considered in Example 4.6. Note that if  $x$  and  $y$  both lie in  $Z_X$ , then  $\varphi$  takes its values in  $Z(A)$  (the center of  $A$ ). Moreover, if  $x = y \in Z_X$ ,

then we have  $\varphi(g) = \langle x, v(g)x \rangle$  for all  $g \in G$ , so  $\varphi$  is of positive type (w.r.t.  $\alpha$ ) in the sense of Anantharaman-Delaroche [1] (assuming  $\sigma$  is trivial). We do not know whether functions from  $G$  to  $A$  of positive type give rise to multipliers of  $\Sigma$  in general.

**Example 4.12** Let  $w$  be a unitary representation of  $G$  on a Hilbert space  $\mathcal{H}$  and let  $(\rho, v)$  be an equivariant representation of  $\Sigma$  on the Hilbert module  $X$ . We can then consider the Hilbert  $A$ -module  $X \otimes \mathcal{H}$  and the equivariant representation  $(\rho \otimes \iota, v \otimes w)$  of  $\Sigma$  on  $X \otimes \mathcal{H}$ . (We leave to the reader to verify that this is indeed an equivariant representation; note that  $(\rho \otimes \iota, v \otimes w)$  will not necessarily give an equivariant representation if  $w$  is assumed to be a uniformly bounded representation of  $G$ ).

Let  $x, y \in X$  and let  $\xi, \eta \in \mathcal{H}$ , so that  $x \otimes \xi, y \otimes \eta \in X \otimes \mathcal{H}$ . Then, by Theorem 4.8, we get a multiplier  $T' \in M_0A(\Sigma)$  given by

$$\begin{aligned} T'(g, a) &= \langle x \otimes \xi, (\rho \otimes \iota)(a)(v \otimes w)(g)(y \otimes \eta) \rangle \\ &= \langle x \otimes \xi, \rho(a)v(g)y \otimes w(g)\eta \rangle \\ &= \langle x, \rho(a)v(g)y \rangle \langle \xi, w(g)\eta \rangle, \quad \text{for all } a \in A \text{ and } g \in G. \end{aligned}$$

Note that if  $(\rho, v) = (\ell, \alpha)$  and  $x = y = 1$ , then  $T'(g, a) = \langle \xi, w(g)\eta \rangle a$ , so  $T' = T^\varphi$  where  $\varphi(g) = \langle \xi, w(g)\eta \rangle$ . Thus  $B(G)$  naturally embeds into  $B(\Sigma)$ .

This example is an illustration that we have  $B(G)B(\Sigma) \subset B(\Sigma)$  (with respect to the natural product structure in  $B(\Sigma)$ ).

**Example 4.13** Let  $(\rho, v)$  be an equivariant representation of  $\Sigma$  on  $X$  and consider the induced regular equivariant representation  $(\check{\rho}, \check{v})$  of  $\Sigma$  on  $H_X^G$ . Theorem 4.8 gives that the map  $(g, a) \rightarrow \langle \xi, \check{\rho}(a)\check{v}(g)\eta \rangle$  is a cb-multiplier of  $\Sigma$  for any  $\xi, \eta \in H_X^G$ .

Note that this fact can also be deduced from [5, Prop. 4.13] (by letting  $(\pi, u)$  in this proposition be  $(\ell_\Sigma, \lambda_\Sigma)$  and using that  $\tilde{\rho} \times \tilde{\lambda}_\rho$  is weakly contained in  $\Lambda_\Sigma$ ).

**Example 4.14** Let  $\beta$  be an endomorphism of  $A$  and assume  $\beta$  satisfies the following two conditions:

- (i)  $\beta\alpha_g = \alpha_g\beta$  for all  $g \in G$ ,
- (ii)  $\beta(\sigma(g, h)) = \sigma(g, h)$  for all  $g, h \in G$ .

Then one checks easily that  $\beta$  extends to an endomorphism  $\tilde{\beta}$  of  $\Lambda_\Sigma(C_c(\Sigma))$  satisfying

$$\tilde{\beta}(a\lambda_\Sigma(g)) = \beta(a)\lambda_\Sigma(g), \quad \text{for all } a \in A \text{ and } g \in G. \quad (21)$$

To show that  $\beta$  extends to an endomorphism of  $C_r^*(\Sigma)$ , we can consider the equivariant representation of  $\Sigma$  on  $A$  given by  $(\rho_\beta, \alpha)$ , where  $\rho_\beta(a)b = \beta(a)b$ , for  $a, b \in A$ . We leave it as an exercise to verify that conditions (i) and (ii) imply that this is indeed an equivariant representation. Choosing  $x = y = 1 \in A$  gives  $\langle x, \rho_\beta(a)\alpha(g)y \rangle = \beta(a)$  for all  $a \in A$  and  $g \in G$ , so Theorem 4.8 tells us that there exists a cb-map  $T_\beta$  on  $C_r^*(\Sigma)$  satisfying Eq. (21). Since  $T_\beta$  coincides with  $\tilde{\beta}$  on  $\Lambda_\Sigma(C_c(\Sigma))$ , it follows that  $T_\beta$  is an endomorphism of  $C_r^*(\Sigma)$  extending  $\beta$ , as desired.

One of our motivations for studying multipliers is that they naturally appear in the context of summation processes for Fourier series of elements in  $C_r^*(\Sigma)$ , that we will discuss in the next section. As in [4], which deals with the case where  $A = \mathbb{C}$ , we will be interested in multipliers that have some kind of smoothing property.

To explain this, consider  $T \in MA(\Sigma)$  and  $x \in C_r^*(\Sigma)$ . Recall that we have

$$\widehat{M_T(x)}(g) = T_g(\widehat{x}(g)), \quad \text{for all } g \in G. \quad (22)$$

This means that the Fourier series of  $M_T(x)$  is

$$\sum_{g \in G} T_g(\widehat{x}(g)) \lambda_\Sigma(g).$$

In general, there is no reason why this series should converge w.r.t. the operator norm for all  $x$  in  $C_r^*(\Sigma)$ , i.e., it may happen that  $M_T(x) \notin CF(\Sigma)$  for some  $x \in C_r^*(\Sigma)$ . We therefore define

$$MCF(\Sigma) = \{T \in MA(\Sigma) \mid M_T(x) \in CF(\Sigma) \text{ for all } x \in C_r^*(\Sigma)\}.$$

Following the proof of [4, Prop. 4.7], one can check that  $MCF(\Sigma)$  consists of all maps  $T : G \times A \rightarrow A$  that are linear in the second variable and satisfy that the series  $\sum_{g \in G} T_g(\widehat{x}(g)) \lambda_\Sigma(g)$  converges w.r.t.  $\|\cdot\|$  for every  $x \in C_r^*(\Sigma)$ .

Of course, if  $T \in MA(\Sigma)$  has *finite  $G$ -support*, that is,  $T_g = 0$  for all but finitely many  $g$ 's in  $G$ , then the Fourier series of  $M_T(x)$  is just a finite sum for every  $x \in C_r^*(\Sigma)$ , so  $T \in MCF(\Sigma)$ . But one can easily find examples without finite  $G$ -support:

**Example 4.15** Let  $\varphi \in \ell^1(G)$ . As  $\ell^1(G) \subset \ell^2(G) \subset B(G)$ ,  $T^\varphi \in M_0A(\Sigma)$ . Moreover,

$$\widehat{M_{T^\varphi}(x)} = \widehat{M_\varphi(x)} = \varphi \widehat{x} \in \ell^1(G, A)$$

for all  $x \in C_r^*(\Sigma)$ , so  $M_{T^\varphi}(x) \in CF(\Sigma)$  for all  $x$ , hence  $T^\varphi \in MCF(\Sigma)$ .

When  $A = \mathbb{C}$ , it is not difficult to show that  $T^\varphi \in MCF(\Sigma)$  whenever  $\varphi \in \ell^2(G)$ , cf. [4, Sect. 4, p. 356]. But the argument given there does not carry over to the general case, and we do not know if this assertion always holds when  $A$  is non-trivial.

The next proposition shows how multipliers belonging to  $MCF(\Sigma)$  may be produced in a way similar to [26, Lemma 1.7] and [4, Prop. 4.8]. It explains why the  $H_\kappa^\Sigma$ -spaces introduced in Sect. 3 have to be taken into consideration.

**Proposition 4.16** *Let  $\kappa : G \rightarrow [1, \infty)$  and assume that  $\Sigma$  has the  $H_\kappa^\Sigma$ -decay property with decay constant  $C$ .*

*Let  $\psi \in \ell_\kappa^\infty(G, A)$ , that is,  $\psi : G \rightarrow A$  satisfies  $K = \|\psi\kappa\|_\infty = \sup_{g \in G} \|\psi(g)\kappa(g)\| < \infty$ .*

*Let  $L^\psi : G \times A \rightarrow A$  be given by  $L^\psi(g, a) = \psi(g)a$ .*

*Then  $L^\psi \in MCF(\Sigma)$  with  $\|L^\psi\| \leq CK$ .*

*Proof* Let  $\xi \in H^\Sigma$ . Then

$$\begin{aligned} \|\psi \xi\|_{\alpha, \kappa}^2 &= \|\psi \xi \kappa\|_\alpha^2 = \left\| \sum_{g \in G} \alpha_g^{-1} (\xi(g)^* \psi(g)^* \psi(g) \xi(g)) \kappa(g)^2 \right\| \\ &\leq \left\| \sum_{g \in G} \alpha_g^{-1} (\xi(g)^* \xi(g)) \|\psi \kappa(g)\|^2 \right\| \\ &\leq K^2 \left\| \sum_{g \in G} \alpha_g^{-1} (\xi(g)^* \xi(g)) \right\| = K^2 \|\xi\|_\alpha^2. \end{aligned}$$

Hence, for any  $f \in C_c(\Sigma)$ , using (8), we have

$$\|M_{L^\psi}(\Lambda_\Sigma(f))\| = \|\Lambda_\Sigma(\psi f)\| \leq C \|\psi f\|_{\alpha, \kappa} \leq CK \|f\|_\alpha \leq CK \|\Lambda_\Sigma(f)\|.$$

This shows that  $L^\psi \in MA(\Sigma)$  with  $\|L^\psi\| \leq CK$ . Let  $x \in C_r^*(\Sigma)$ . Then  $\widehat{x} \in H^\Sigma$ , so the above computation gives that

$$\|\psi \widehat{x}\|_{\alpha, \kappa} \leq K \|\widehat{x}\|_\alpha < \infty.$$

Thus  $\widehat{M_{L^\psi}(x)} = \psi \widehat{x} \in H_\kappa^\Sigma$ , and it follows from Proposition 3.2 that  $M_{L^\psi}(x) \in C F(\Sigma)$ .  $\square$

It seems very unlikely to us that Proposition 4.16 remains true in general if we replace  $H_\kappa^\Sigma$ -decay with  $\ell_\kappa^2(G, A)$ -decay in the assumption.

## 5 Summation Processes for Fourier Series

By a *Fourier summing net* for  $\Sigma$ , we will mean a net  $\{T^i\}$  in  $MC F(\Sigma)$  such that

$$\lim_i \|M_{T^i}(x) - x\| = 0 \quad \text{for all } x \in C_r^*(\Sigma). \quad (23)$$

We will say that such a net is *bounded* whenever  $\sup_i \|T^i\| < \infty$ . We will repeatedly use the fact that, in order to show that a net  $\{T^i\}$  in  $MC F(\Sigma)$  is a bounded Fourier summing net, one only needs to check that

$$\sup_i \|T^i\| < \infty \text{ and } \lim_i T_g^i(a) = a \text{ for all } g \in G \text{ and } a \in A. \quad (24)$$

This assertion is easily shown using an  $\varepsilon/3$ -argument.

Assume that  $\{T^i\}$  is a Fourier summing net for  $\Sigma$ . Note that, as each  $T^i$  is assumed to lie in  $MC F(\Sigma)$ , the series

$$\sum_{g \in G} T_g^i(\widehat{x}(g)) \lambda_\Sigma(g)$$

is convergent in operator norm for each  $x \in C_r^*(\Sigma)$  and each  $i$ , and Eq. (23) gives

$$\lim_i \sum_{g \in G} T_g^i(\widehat{x}(g)) \lambda_\Sigma(g) = x \quad \text{for all } x \in C_r^*(\Sigma)$$

with respect to the operator norm on  $C_r^*(\Sigma)$ . Hence, a Fourier summing net  $\{T^i\}$  for  $\Sigma$  provides a summation process for the Fourier series of all elements in  $C_r^*(\Sigma)$ .

We will also be interested in Fourier summing nets satisfying an additional property: A Fourier summing net  $\{T^i\}$  for  $\Sigma$  will be said to *preserve the invariant ideals of  $A$*  if every invariant ideal of  $A$  is preserved by each  $T_g^i$ , that is, for every invariant ideal  $J$  of  $A$  we have

$$T_g^i(J) \subset J \quad \text{for every } i \text{ and every } g \in G.$$

Of course, by an invariant ideal of  $A$  we mean as usual an ideal of  $A$  left invariant by each  $\alpha_g$ . As we will discuss below, the existence of a Fourier summing net for  $\Sigma$  that preserves the invariant ideals of  $A$  has some useful consequences when studying the ideal structure of  $C_r^*(\Sigma)$ . This was first observed by Zeller-Meier when  $G$  is amenable (cf. [48, Prop. 5.10]), and by R. Exel [19, Prop. 4.10] when  $\Sigma$  has the approximation property (in the setting of Fell bundles). From a purely  $C^*$ -algebraic point of view, these nets are those of primary interest. However, as we will soon see, Fourier summing nets preserving the invariant ideals do not necessarily exist when  $G$  is not *exact*, that is,  $C_r^*(G)$  is not exact as a  $C^*$ -algebra (see [9, Sect. 5.1]).

We recall some more terminology and introduce some notation.

Let  $J$  be an invariant ideal of  $A$ . The ideal of  $C_r^*(\Sigma)$  generated by  $J$  will be denoted by  $\langle J \rangle$  and called an *induced ideal* of  $C_r^*(\Sigma)$ . Moreover,  $q : A \rightarrow \tilde{A} = A/J$  will denote the quotient map,  $\tilde{\Sigma} = (\tilde{A}, G, \tilde{\alpha}, \tilde{\sigma})$  the induced quotient system (defined in the obvious way) and  $\tilde{q}$  the canonical homomorphism from  $C_r^*(\Sigma)$  onto  $C_r^*(\tilde{\Sigma})$ , determined by  $\tilde{q} \Lambda_\Sigma = \Lambda_{\tilde{\Sigma}} q$ . Then we set  $\tilde{J} = \text{Ker } \tilde{q}$ . Finally, we set

$$J^\vee = \{x \in C_r^*(\Sigma) \mid \widehat{x}(g) \in J \text{ for all } g \in G\}.$$

**Proposition 5.1** *Let  $J$  be an invariant ideal of  $A$ . Then we have*

$$\langle J \rangle \subset \tilde{J} \subset J^\vee.$$

*Assume that there exists a Fourier summing net  $\{T^i\}$  for  $\Sigma$  that preserves  $J$ . Then we have*

$$\langle J \rangle = \tilde{J} = J^\vee.$$

*Proof* The first inclusion is well known, at least when  $\sigma$  is trivial. For completeness, we sketch the argument. Using the invariance of  $J$  and the covariance relation, one

sees that  $\langle J \rangle$  is the norm closure of

$$\langle J \rangle_{alg} = \left\{ \sum_{g \in F} a_g \lambda_\Sigma(g) \mid F \subset G, F \text{ finite}, a_g \in J \text{ for all } g \in F \right\}.$$

As  $\tilde{q}$  obviously maps  $\langle J \rangle_{alg}$  to  $\{0\}$  and  $\tilde{J}$  is closed, it is clear that  $\langle J \rangle \subset \tilde{J}$ .

Next, it is an easy exercise to check that for all  $x \in C_r^*(\Sigma)$  and  $g \in G$ , we have

$$q(\widehat{x}(g)) = \widehat{q(x)}(g).$$

It follows that if  $x \in \tilde{J}$ , then  $q(\widehat{x}(g)) = 0$  for every  $g \in G$ , hence that  $\widehat{x}(g) \in J$  for every  $g \in G$ . This shows the second inclusion.

Now, assume that there exists a Fourier summing net  $\{T^i\}$  for  $\Sigma$  that preserves  $J$  and consider  $x \in J^\vee$ . For every  $i$ , set

$$x_i = \sum_{g \in G} T_g^i(\widehat{x}(g)) \lambda_\Sigma(g).$$

Using the assumption, we have  $T_g^i(\widehat{x}(g)) \in J$  for every  $i$  and every  $g \in G$ . As  $\langle J \rangle$  is closed, we get  $x_i \in \langle J \rangle$  for every  $i$ . Since  $x$  is the norm-limit of  $\{x_i\}$ , this implies that  $x \in \langle J \rangle$ . This shows that  $J^\vee \subset \langle J \rangle$  and the last assertion clearly follows.  $\square$

It can be shown that the equality  $\tilde{J} = J^\vee$  always holds, and that if  $G$  is exact, then we also have  $\langle J \rangle = \tilde{J} = J^\vee$  for every invariant ideal  $J$  of  $A$ . For a proof in the setting of Fell bundles over discrete groups, we refer to [20, Theorem 5.2]<sup>1</sup> and its proof.

We will say that  $\Sigma = (A, G, \alpha, \sigma)$  is *exact* whenever we have  $\langle J \rangle = \tilde{J}$  for every invariant ideal  $J$  of  $A$ . When  $\sigma$  is trivial, this terminology was recently introduced by Sierakowski in [43] to give a characterization of systems  $(A, G, \alpha)$  having the property that all ideals of  $C_r^*(A, G, \alpha)$  are induced. As shown by Kirchberg and Wassermann (cf. [9, Theorem 5.1.10]), it is then known that  $G$  is exact if and only if  $(B, G, \beta)$  is exact for every action  $\beta$  of  $G$  on some  $C^*$ -algebra  $B$ . In fact,  $G$  is exact if and only if the system  $(B, G, \beta, \omega)$  is exact for every twisted action  $(\beta, \omega)$  of  $G$  on a  $C^*$ -algebra  $B$ , as follows easily from [21, Theorem 4.4]. Now, an immediate consequence of Proposition 5.1 is:

**Corollary 5.2** *Assume that there exists a Fourier summing net  $\{T^i\}$  for  $\Sigma$  that preserves the invariant ideals of  $A$ . Then  $\Sigma$  is exact.*

One may therefore wonder whether exactness of  $G$  always implies the existence of a Fourier summing net for  $\Sigma$  that preserves the invariant ideals of  $A$ .

On the other hand, assume that  $G$  is not exact. This means that there exists a  $C^*$ -algebra  $B$  such that  $(B, G, \text{id})$  is not exact (since  $C_r^*(B, G, \text{id}) \simeq B \otimes C_r^*(G)$ ). Hence, it follows from Corollary 5.2 that there exists no Fourier summing net for  $(B, G, \text{id})$  that preserves the (invariant) ideals of  $B$ .

<sup>1</sup> This article of R. Exel is the preprint version of [21]. It contains a section on induced ideals that was removed in the published version.



It is also known that  $G$  is exact if and only if  $C_r^*(B, G, \beta, \omega)$  is exact for every twisted action  $(\beta, \omega)$  of  $G$  on an exact  $C^*$ -algebra  $B$  (see [2, Theorem 7.2 and Remark 7.4] for the case of untwisted actions; the twisted case can be handled in a similar way). However, if  $A$  is exact and  $\Sigma$  is exact, then  $C_r^*(\Sigma)$  is not necessarily exact: to see this, one may for instance consider the trivial action of a non-exact group on a simple exact  $C^*$ -algebra. The following result is therefore noteworthy:

**Proposition 5.3** *Assume that there exists a Fourier summing net  $\{T^i\}$  for  $\Sigma$  that preserves the invariant ideals of  $A$ . Then  $C_r^*(\Sigma)$  is exact if and only if  $A$  is exact.*

*Proof* We only sketch the proof, as it is close to the proof of [2, Theorem 7.2]. Assume that  $A$  is exact and let  $0 \rightarrow I \rightarrow B \rightarrow B/I \rightarrow 0$  be a short exact sequence for some  $C^*$ -algebra  $B$ . Consider the twisted action  $(\alpha \otimes \text{id}_B, \sigma \otimes 1_B)$  of  $G$  on  $A \otimes B$ .

For each  $i$ , let  $S^i : G \times (A \otimes B) \rightarrow A \otimes B$  be given by  $S_g^i = T_g^i \otimes \text{id}_B$  for each  $g \in G$ . Then it is easy to check that  $\{S^i\}$  is a Fourier summing net for the system  $\Omega = (A \otimes B, G, \alpha \otimes \text{id}_B, \sigma \otimes 1_B)$ . Now  $J = A \otimes I$  is an invariant ideal of  $A \otimes B$  that is clearly preserved by  $\{S^i\}$ . Proposition 5.1 gives therefore that  $\langle J \rangle = \tilde{J}$ . Using the obvious identification of  $C_r^*(\Omega)$  with  $C_r^*(\Sigma) \otimes B$ , one then observes that this fact corresponds to the exactness of the sequence

$$0 \rightarrow C_r^*(\Sigma) \otimes I \rightarrow C_r^*(\Sigma) \otimes B \rightarrow C_r^*(\Sigma) \otimes B/I \rightarrow 0.$$

This shows that  $C_r^*(\Sigma)$  is exact. The converse implication is trivial since exactness of  $C^*$ -algebras passes to  $C^*$ -subalgebras.  $\square$

We will show below that if there exists a Fourier summing net for  $\Sigma$  that preserves the invariant ideals of  $A$ , then the induced ideals of  $C_r^*(\Sigma)$  may be characterized by certain invariance properties.

Let  $\mathcal{J}$  be an ideal of  $C_r^*(\Sigma)$ . Then  $\mathcal{J} \cap A$  is an invariant ideal of  $A$ , that may be equal to  $\{0\}$  even if  $\mathcal{J} \neq \{0\}$ . On the other hand,  $E(\mathcal{J})$  is easily seen to be an invariant algebraic ideal of  $A$  that contains  $\mathcal{J} \cap A$ . Moreover, since  $E$  is faithful, we have  $E(\mathcal{J}) \neq \{0\}$  if  $\mathcal{J} \neq \{0\}$ . However, it is not obvious that  $E(\mathcal{J})$  is necessarily closed in general.

We will say that  $\mathcal{J}$  is *E-invariant* when  $E(\mathcal{J}) \subset \mathcal{J}$ .

Note that when  $G = \mathbb{Z}$  and  $\sigma$  is trivial, *E*-invariant ideals of  $C_r^*(\Sigma)$  are called *well behaving* in [46]. It is straightforward to see that an ideal  $\mathcal{J}$  of  $C_r^*(\Sigma)$  is *E*-invariant if and only if  $E(\mathcal{J}) = \mathcal{J} \cap A$ , in which case  $E(\mathcal{J})$  is then a (closed) invariant ideal of  $A$ .

It is well known and easy to check that any induced ideal of  $C_r^*(\Sigma)$  is *E*-invariant. It is not known in general whether the converse is true, i.e., whether any *E*-invariant ideal of  $C_r^*(\Sigma)$  is induced. However, this holds whenever  $G$  is exact, as shown by Exel [20, Corollary 5.3].

The concept of *E*-invariance is related to another kind of invariance. Following [24, 36], we will say that an ideal  $\mathcal{J}$  of  $C_r^*(\Sigma)$  is  $\delta_\Sigma$ -invariant whenever

$$\delta_\Sigma(\mathcal{J}) \subset \mathcal{J} \otimes C_r^*(G).$$

Here  $\delta_\Sigma$  denotes the (reduced) dual coaction of  $G$  on  $\Sigma$  defined in Sect. 4. It is evident that every induced ideal of  $C_r^*(\Sigma)$  is  $\delta_\Sigma$ -invariant. Moreover, every  $\delta_\Sigma$ -invariant ideal of  $C_r^*(\Sigma)$  is  $E$ -invariant: indeed, this follows readily after checking that we have  $E = (\text{id} \otimes \tau)\delta_\Sigma$ , where  $\tau$  denotes the canonical tracial state on  $C_r^*(G)$ .

Hence, if  $G$  is exact, we get from Exel's result mentioned above that an ideal  $\mathcal{I}$  of  $C_r^*(\Sigma)$  is  $E$ -invariant if and only if it is  $\delta_\Sigma$ -invariant, if and only if it is induced. In the case where  $G$  is amenable and  $\sigma$  is trivial, the last part of this statement follows from [24, Theorem 3.4].

**Proposition 5.4** *Assume that  $G$  is exact or that there exists a Fourier summing net  $\{T^i\}$  for  $\Sigma$  that preserves the invariant ideals of  $A$ .*

*Then an ideal of  $C_r^*(\Sigma)$  is  $E$ -invariant if and only if it is  $\delta_\Sigma$ -invariant, if and only if it is induced.*

*Hence, the map  $J \rightarrow \langle J \rangle$  is a bijection between the set of all invariant ideals of  $A$  and the set of all  $E$ -invariant ideals of  $C_r^*(\Sigma)$ .*

*Proof* As the map  $J \rightarrow \langle J \rangle$  is injective, the second assertion will follow immediately from the first. Moreover, we have just seen that the first assertion holds whenever  $G$  is exact. Hence, we assume that there exists a Fourier summing net  $\{T^i\}$  for  $\Sigma$  that preserves the invariant ideals of  $A$ . To show that the first assertion holds in this case, in view of our considerations above, it suffices to prove that every  $E$ -invariant ideal of  $C_r^*(\Sigma)$  is induced. So let  $\mathcal{I}$  be an  $E$ -invariant ideal of  $C_r^*(\Sigma)$  and set  $J = \mathcal{I} \cap A$ , i.e.  $J = E(\mathcal{I})$ . Note that

$$\langle J \rangle \subset \mathcal{I} \subset J^\vee.$$

Indeed, the first inclusion is immediate since  $J$  is contained in the ideal  $\mathcal{I}$ . Now, let  $x \in \mathcal{I}$  and  $g \in G$ . Then  $x \lambda_\Sigma(g)^* \in \mathcal{I}$ , so

$$\widehat{x}(g) = E(x \lambda_\Sigma(g)^*) \in E(\mathcal{I}) = J.$$

This shows that  $x \in J^\vee$  and the second inclusion follows. Appealing to Proposition 5.1, we can then conclude that  $\mathcal{I} = \langle J \rangle = J^\vee$ , hence that  $\mathcal{I}$  is induced, as desired.  $\square$

**Example 5.5** Assume that  $G$  is a weak Powers group (see [3, 29] and references therein), e.g.  $G$  is a non-abelian free group or a free product of non-trivial groups that is different from  $\mathbb{Z}_2 * \mathbb{Z}_2$ . We recall a few facts from [3]. A simple  $G$ -averaging process on  $C_r^*(\Sigma)$  is a map  $\phi$  from  $C_r^*(\Sigma)$  into itself such that for some  $n \in \mathbb{N}$  and  $s_1, \dots, s_n \in G$  we have

$$\phi(x) = \frac{1}{n} \sum_{i=1}^n \lambda_\Sigma(s_i) x \lambda_\Sigma(s_i)^* \quad \text{for all } x \in C_r^*(\Sigma).$$

A  $G$ -averaging process  $\psi$  is a composition of finitely many simple  $G$ -averaging processes. Note that such a linear map  $\psi$  is positive and maps any ideal of  $C_r^*(\Sigma)$  into itself. Lemma 4.6 in [3] says that if  $x^* = x \in C_r^*(\Sigma)$  is given, then for every  $\varepsilon > 0$

there exists a  $G$ -averaging process  $\psi_\varepsilon$  such that  $\|\psi_\varepsilon(x - E(x))\| < \varepsilon$ . This lemma is used in [3] to show that  $C_r^*(\Sigma)$  is simple whenever  $A$  is  $\alpha$ -simple, i.e.,  $A$  has no other invariant ideals than  $\{0\}$  and  $A$  itself. (This result was first proved by de la Harpe and Skandalis [30] when  $G$  is a Powers group and  $\sigma$  is trivial).

Let us now assume that there exists a Fourier summing net for  $\Sigma$  that preserves the invariant ideals of  $A$  (or that  $G$  is exact). We then know from Proposition 5.4 that the invariant ideals of  $A$  are in a one-to-one correspondence with the  $E$ -invariant ideals of  $C_r^*(\Sigma)$ .

We consider first the case where  $\alpha$  is trivial. As any  $G$ -averaging process then restricts to the identity map on  $A$ , it clearly follows from the lemma cited above that any ideal of  $C_r^*(\Sigma)$  is  $E$ -invariant in this case. Hence, we get that the ideals of  $A$  are in a one-to-one correspondence with the ideals of  $C_r^*(\Sigma)$ . We note that if we also assume that  $A$  is commutative, then the existence of a Fourier summing net that preserves the ideals of  $A$  may be deduced from Corollary 6.5 in certain cases (see Example 6.6).

When  $\alpha$  is not trivial, the ideal structure of  $C_r^*(\Sigma)$  can be much more complicated. Nevertheless, we can obtain some valuable information: as we will show below, the map  $J \rightarrow \langle J \rangle$  gives a bijection between the maximal invariant ideals of  $A$  and the maximal ideals of  $B = C_r^*(\Sigma)$ .

If  $J$  is a maximal invariant ideal of  $A$  (such an ideal must exist by Zornification), then, using the same notation as in Proposition 5.1,  $B/\langle J \rangle \simeq C_r^*(\tilde{A}, G, \tilde{\alpha}, \tilde{\sigma})$  is simple since  $\tilde{A} = A/J$  is  $\tilde{\alpha}$ -simple (and  $G$  is a weak Powers group). Hence,  $\langle J \rangle$  is maximal in  $B$ .

Next, let  $\mathcal{J}$  be a *proper* ideal of  $B$ . Then  $J = \overline{E(\mathcal{J})}$  is a *proper* ideal of  $A$ . Indeed, assume (by contradiction) that  $J = A$ . Then, as  $A$  is unital,  $E(\mathcal{J}) = A$ . So pick  $x \in \mathcal{J}$  such that  $E(x) = 1$ . Then, as  $E$  is a Schwarz map (cf. [9, Prop. 1.5]),  $E(x^*x) \geq E(x)^*E(x) = 1$ . Thus  $y = x^*x \in \mathcal{J}^+$  satisfies  $E(y) \geq 1$ . Using the lemma cited above, we can find a  $G$ -averaging process  $\psi$  such that

$$\|\psi(y) - \psi(E(y))\| < \frac{1}{2}.$$

Since  $\psi(E(y)) \geq \psi(1) = 1$  and  $\psi(y)$  is positive, this implies that  $\psi(y) \in \mathcal{J}$  is invertible. Hence  $\mathcal{J} = B$ , contradicting that  $\mathcal{J}$  is proper.

Moreover,  $J$  is clearly invariant and satisfies  $\mathcal{J} \subset J^\vee$ . Using Proposition 5.1 (or the remark following it if  $G$  is exact), we have  $J^\vee = \langle J \rangle$ , hence  $\mathcal{J} \subset \langle J \rangle$ .

Now, if  $\mathcal{J}$  is assumed to be maximal, then we get  $\mathcal{J} = \langle J \rangle$  and  $J$  is necessarily maximal among the invariant ideals of  $A$ . This proves our assertion.

Following [4], we introduce some more terminology. We will say that  $\Sigma$  has *the Fejér property* if there exists a Fourier summing net  $\{T^i\}$  for  $\Sigma$  such that each  $T^i$  has finite  $G$ -support. If such a net  $\{T^i\}$  can be chosen to be bounded,  $\Sigma$  will be said to have *the bounded Fejér property*. It is a well-known result due to Zeller-Meier [48, Prop. 5.10] that  $\Sigma$  has the bounded Fejér property whenever  $G$  is amenable and  $\sigma$  is central. (See [13, Theorem VIII.2.2] for a short proof in the case where  $G = \mathbb{Z}$  and  $\sigma$  is trivial; this case is also discussed in [45]). The direct analogue of Fejér's classical summation theorem for twisted group  $C^*$ -algebras of amenable groups [4, Theorem 5.6] is still valid in our more general setting:

**Theorem 5.6** Assume  $G$  is amenable. Then  $\Sigma$  has the bounded Fejér property. Indeed, pick a Følner net  $\{F_i\}$  for  $G$  and let  $T^i : G \times A \rightarrow A$  be given by

$$T^i(g, a) = \frac{|gF_i \cap F_i|}{|F_i|}a, \quad \text{for all } g \in G \text{ and } a \in A.$$

Then  $\{T^i\}$  is a Fourier summing net for  $\Sigma$  such that each  $T^i$  has finite  $G$ -support and  $\|T^i\| = 1$  for each  $i$ .

*Proof* As in [4], set  $\varphi_i(g) = \frac{|gF_i \cap F_i|}{|F_i|}$  for  $g \in G$ . Then  $\{\varphi_i\}$  is a net in  $C_c(G)$  of normalized positive definite functions converging pointwise to 1. As  $T^i = T^{\varphi_i}$ , each  $T^i$  has finite  $G$ -support and it follows from Corollary 4.3 that  $T^i \in MCF(\Sigma)$  with  $\|T^i\| = \varphi_i(e) = 1$  for each  $i$ .  $\square$

Theorem 5.6 may be generalized to a class of groups containing nonamenable groups. We recall from [9, Sect. 12.3] that  $G$  is called *weakly amenable* if there exists a net  $\{\varphi_i\}$  of finitely supported scalar-valued functions on  $G$  which converges pointwise to 1 and is bounded, that is,  $\sup_i \|M_{\varphi_i}\|_{cb} < \infty$ , where  $M_{\varphi_i} : C_r^*(G) \rightarrow C_r^*(G)$  denotes the completely bounded map associated to each  $\varphi_i$ . The class of weakly amenable groups contains for example all amenable groups and all groups acting properly on a tree. It is closed under taking subgroups and Cartesian products. See [9] for other examples.

**Theorem 5.7** Assume  $G$  is weakly amenable. Then  $\Sigma$  has the bounded Fejér property.

*Proof* This is a direct consequence of Corollary 4.7.  $\square$

When  $G$  is weakly amenable, it is easy to see that the Fourier summing net for  $\Sigma$  produced in the proof of Theorem 5.7 will preserve the invariant ideals of  $A$ , so Proposition 5.4 may be applied. Alternatively, one could use that  $G$  is then known to be exact. In fact, if  $G$  is weakly amenable, then  $G$  has Haagerup's and Kraus' approximation property AP [27], and if  $G$  has the AP, then  $G$  is exact (see [9, Sect. 12.4]). Note that there are groups having the AP without being weakly amenable [9, p. 373], and that it follows from the recent work of Lafforgue and de la Salle [34] (see also [28]) that there are examples of exact groups without the AP. In this connection, it would be interesting to know whether  $\Sigma$  will have the Fejér property whenever  $G$  has the AP, or more generally, whenever  $G$  is exact.

Instead of conditions involving only the group  $G$ , one may look for conditions on  $\Sigma$ . In this direction, we have:

**Theorem 5.8** Assume that  $\Sigma$  has the weak approximation property. Then  $\Sigma$  has the bounded Fejér property.

Moreover, assume that  $\Sigma$  has the approximation property, or the half-central weak approximation property.

Then  $\Sigma$  is exact, while  $C_r^*(\Sigma)$  is exact if and only if  $A$  is exact. We also have that the  $E$ -invariant ideals of  $C_r^*(\Sigma)$  are in a one-to-one correspondence with the invariant ideals of  $A$ .

*Proof* Let  $X$ ,  $(\rho, v)$ ,  $M$ ,  $\{\xi_i\}$  and  $\{\eta_i\}$  be as in the definition of the weak approximation property, each  $\xi_i$  (resp.  $\eta_i$ ) being chosen in  $H_X^G$  with finite support  $\text{supp}(\xi_i)$  (resp.  $\text{supp}(\eta_i)$ ).

For each  $i$ , define  $T^i : G \times A \rightarrow A$  by  $T^i(g, a) = \langle \xi_i, \check{\rho}(a)\check{v}(g)\eta_i \rangle$ , that is,

$$T^i(g, a) = \sum_{h \in G} \langle \xi_i(h), \rho(a) v(g)\eta_i(g^{-1}h) \rangle, \quad \text{for all } g \in G \text{ and } a \in G. \quad (25)$$

From Theorem 4.8, see also Example 4.13, we know that  $T^i \in MA(\Sigma)$  and satisfies  $\|T^i\| \leq \|\xi_i\| \|\eta_i\|$  for each  $i$ . Since  $\|\xi_i\| \|\eta_i\| \leq M$  for each  $i$ , we see that  $\{T^i\}$  is bounded.

Moreover, we have  $\lim_i \|T^i(g, a) - a\| = 0$  for each  $g \in G$  and  $a \in A$  by assumption. Finally, Eq. (25) gives that each  $T^i$  has finite  $G$ -support equal to  $\text{supp}(\xi_i) \cdot (\text{supp}(\eta_i))^{-1}$ . Altogether, this shows that  $\{T^i\}$  is a bounded Fourier summing net for  $\Sigma$  such that each  $T^i$  has finite  $G$ -support, and the first assertion is proven.

Now, assume first that  $\Sigma$  has the half-central weak approximation property, which means that  $\{\xi_i\}$  or  $\{\eta_i\}$  may be chosen to lie in the central part of  $H_X^G$ . As shown in Example 4.11, each  $T^i$  is then a multiplier obtained by multiplication (from the left or from the right) with a function from  $G$  to  $A$ . It is therefore obvious that  $\{T^i\}$  preserves (all) ideals of  $A$ .

Next, assume that  $\Sigma$  has the approximation property, that is, we have  $(\rho, v) = (\ell, \alpha)$  and  $\{\xi_i\}, \{\eta_i\} \subset H_A^G$ . Then, for every  $i$  and every  $g \in G, a \in A$ , we have

$$T_g^i(a) = \sum_{h \in G} \xi_i(h)^* a \alpha_g(\eta_i(g^{-1}h)),$$

and it is evident that  $\{T^i\}$  preserves (all) ideals of  $A$  also in this case.

Hence, the second part of the theorem follows from Corollary 5.2, Proposition 5.3 and Proposition 5.4.  $\square$

Note that when  $\Sigma$  has the approximation property, the first and the final assertions of Theorem 5.8 are closely related to [19, Props. 4.9 and 4.10], since  $C_r^*(\Sigma)$  may be written as the reduced sectional algebra of a Fell bundle over  $G$  [22]. Examples of systems (with  $\sigma$  trivial) satisfying a strong version of the approximation property (called amenability) may be found in [9, Chaps. 4 and 5] (see also [1, 2]). For such amenable systems, the third assertion of Theorem 5.8 is already known, cf. [9, Theorem 4.3.4, part (3)].

*Example 5.9* Assume that  $G$  is exact,  $H$  is an amenable subgroup of  $G$ ,  $A = \ell^\infty(G/H)$ ,  $\alpha$  is the natural action of  $G$  on  $A$  and  $\sigma$  takes values in  $\mathbb{T}$ . Then it is shown in [5, Example 5.19] that  $\Sigma = (\ell^\infty(G/H), G, \alpha, \sigma)$  has the weak approximation property and one may therefore apply Theorem 5.8 to produce a bounded Fejér

summing net for  $\Sigma$ . Moreover, it can be checked<sup>2</sup> that  $\Sigma$  has the central approximation property and the second part of Theorem 5.8 also applies. Alternatively, one could use here that  $G$  is assumed to be exact.

In [4], we discussed analogs of Abel–Poisson summation of Fourier series in reduced twisted group  $C^*$ -algebras. In  $C_r^*(\Sigma)$ , the only case that is straightforward to handle is when  $G = \mathbb{Z}^n$ . Indeed, similarly to [4, Theorem 5.7], we have:

**Theorem 5.10** *Let  $G = \mathbb{Z}^n$  for some  $n \in \mathbb{N}$ . For  $p \in \{1, 2\}$ , let  $|\cdot|_p$  denote the usual  $p$ -norm on  $G$  and let  $K(\cdot)$  denote either  $|\cdot|_1$ ,  $|\cdot|_2$  or  $|\cdot|_2^2$ .*

*For each  $r \in (0, 1)$ , let  $\varphi_r = r^K$  be the function on  $G$  defined by  $\varphi_r(g) = r^{K(g)}$  and set  $T^r = T^{\varphi_r}$ , so*

$$T^r(g, a) = r^{K(g)}a, \quad \text{for all } g \in G \text{ and } a \in A.$$

*Then  $\{T^r\}_{r \rightarrow 1^-}$  is a bounded Fourier summing net for  $\Sigma$ .*

*Proof* As pointed out in the proof of [4, Theorem 5.7],  $\varphi_r$  is a normalized positive definite function on  $G$  for each  $r \in (0, 1)$ . Hence, Corollary 4.3 gives that  $\{T^r\}$  is a bounded net in  $MA(\Sigma)$ . Moreover, each  $\varphi_r$  lies in  $\ell^1(G)$ , so Example 4.15 gives that  $T^r \in MCF(\Sigma)$  for each  $r \in (0, 1)$ . As  $\varphi_r$  converges pointwise to 1 when  $r \rightarrow 1^-$ , we have

$$\lim_{r \rightarrow 1^-} \|T^r(g, a) - a\| = \lim_{r \rightarrow 1^-} |\varphi_r(g) - 1| \|a\| = 0$$

for each  $g \in G$  and  $a \in A$ . Hence the result follows.  $\square$

To show versions of the Abel–Poisson summation theorem for systems associated with other kind of groups, such as Coxeter groups or Gromov hyperbolic groups, the following result, analogous to [4, Prop. 5.8], might prove to be helpful (as in the case  $A = \mathbb{C}$  discussed in [4, Sect. 5]). We will give an application of it in the next section.

**Proposition 5.11** *Let  $\{\psi_i\}$  be a net of functions from  $G$  to  $A$  converging pointwise to 1 and consider the maps  $L^i : G \times A \rightarrow A$  given by  $L^i(g, a) = \psi_i(g)a$ .*

*Assume that for each  $i$  there exists  $\kappa_i : G \rightarrow [1, \infty)$  such that*

- $\Sigma$  has the  $H_{\kappa_i}^\Sigma$ -decay property with decay constant  $C_i$ ,
- $\psi_i \in \ell_{\kappa_i}^\infty(G, A)$ , so  $K_i = \|\psi_i \kappa_i\|_\infty < \infty$ .

*Then  $\{L^i\} \subset MCF(\Sigma)$ . Moreover, if  $\sup_i C_i K_i < \infty$  or, more generally, if  $\{L^i\}$  is bounded, then  $\{L^i\}$  is a bounded Fourier summing net for  $\Sigma$ .*

*Proof* According to Proposition 4.16, the first two conditions ensure that  $L^i \in MCF(\Sigma)$  for each  $i$ . Moreover, as  $\|L^i\| \leq C_i K_i$  for each  $i$  and  $\lim_i L_g^i(a) = \lim_i \psi_i(g)a = a$  for each  $g \in G$  and  $a \in A$ , the final assertion is clear.  $\square$

<sup>2</sup> One has then to have a closer look at the proof of [5, Prop. 5.15]: using the notation used in this proof, one checks easily that if  $\xi$  lies in the central part of  $H_X^G$ , then  $\xi'$  defined by  $\xi'(g) = \xi(g) + N$ , for  $g \in G$ , lies in the central part of  $H_{X'}^G$ .

Assume  $G$  is a Coxeter group or a Gromov hyperbolic group and let  $K$  denote the algebraic length function on  $G$  associated with some finite set of generators for  $G$ . It is known that  $\{\psi_r\}_{r \in (0,1)}$  with  $\psi_r = r^K$  gives a bounded net in  $M_0A(G)$  (cf. [11, 37]). It therefore follows from Corollary 4.7 that the net  $L^r$  associated with  $\{\psi_r\}$  (as in Proposition 5.11) is a bounded net in  $M_0A(\Sigma)$ , hence in  $MA(\Sigma)$ . In order to apply Proposition 5.11 and deduce that  $\{L^r\}$  is a bounded Fourier summing net for  $\Sigma$ , it suffices to show that  $\Sigma$  has the  $H_{\kappa_r}^\Sigma$ -decay property for each  $r \in (0, 1)$ , where  $\kappa_r = r^{-K}$ . Note that  $G$  is  $\kappa_r$ -decaying (because  $G$  has polynomial H-growth w.r.t.  $K$ , cf. [5, Example 3.12]). However we do not know if the  $H_{\kappa_r}^\Sigma$ -decay property may be deduced from this, except when  $A$  is commutative and  $\alpha$  is trivial (see Corollary 6.4).

We also mention a result closely related to Proposition 5.11:

**Proposition 5.12** *Let  $\{\psi_i\}$  be a net of functions from  $G$  to  $A$  converging pointwise to 1 and consider the maps  $L^i : G \times A \rightarrow A$  given by  $L^i(g, a) = \psi_i(g)a$ .*

*Assume that for each  $i$  the following conditions hold:*

- $L^i \in MA(\Sigma)$  with  $\|L^i\| = 1$ ,
- there exists  $\kappa_i : G \rightarrow [1, \infty)$  such that  $\Sigma$  has the  $H_{\kappa_i}^\Sigma$ -decay property and  $\psi_i \kappa_i \in c_0(G, A)$ .

*Then  $\Sigma$  has the bounded Fejér property.*

*Proof* The proof is a verbatim adaptation of the proof of [4, Theorem 7.1] (that itself is an adaptation of [26, Theorem 1.8]), now appealing to Proposition 4.16 instead of invoking [4, Prop. 4.8].  $\square$

Note that if we stick to normalized (scalar-valued) positive definite functions  $\psi_i$  on  $G$  in the above assumptions, then  $G$  must have the Haagerup property (see [11] or [9, Sect. 12.2]). But allowing  $A$ -valued functions might be useful to handle other kind of situations.

## 6 The Almost Trivial Case

In this final section, we take up the issue of finding examples of weight functions  $\kappa$  on  $G$  such that  $\Sigma$  has the  $H_\kappa^\Sigma$ -property in the “almost trivial” case where  $A = C(X)$  is commutative and  $\alpha$  is trivial. In such a situation,  $C_r^*(\Sigma) = C_r^*(C(X), G, \text{id}, \sigma)$  is a (unital discrete) reduced central twisted transformation group algebra, and the variety of  $C^*$ -algebras contained in this class is larger than one might imagine at a first thought; see for example [18] and note that any twisted reduced group  $C^*$ -algebra associated with a central group extension belongs to this class.

We will use the following notation.

For  $a \in A$  and  $\omega \in S(A)$  (the state space of  $A$ ), we set  $\|a\|_\omega = \omega(a^*a)^{1/2}$ .

Let  $\xi \in H_A^G$ . For each  $\omega \in S(A)$ , we define  $|\xi|_\omega : G \rightarrow [0, \infty)$  by

$$|\xi|_\omega(g) = \|\xi(g)\|_\omega = \omega(\xi(g)^*\xi(g))^{1/2}, \quad \text{for all } g \in G.$$

Note that  $|\xi|_\omega \in \ell^2(G)$  since

$$\| |\xi|_\omega \|_2^2 = \sum_{g \in G} \omega(\xi(g)^* \xi(g)) = \omega \left( \sum_{g \in G} \xi(g)^* \xi(g) \right) < \infty.$$

Letting  $\|\xi\|$  denote the norm of  $\xi$  in  $H_A^G$ , we have

$$\begin{aligned} \|\xi\| &= \left\| \sum_{g \in G} \xi(g)^* \xi(g) \right\|^{1/2} = \sup_{\omega \in S(A)} \omega \left( \sum_{g \in G} \xi(g)^* \xi(g) \right)^{1/2} \\ &= \sup_{\omega \in S(A)} \left( \sum_{g \in G} \omega(\xi(g)^* \xi(g)) \right)^{1/2} \\ &= \sup_{\omega \in S(A)} \left( \sum_{g \in G} \|\xi(g)\|_\omega^2 \right)^{1/2} = \sup_{\omega \in S(A)} \| |\xi|_\omega \|_2. \end{aligned}$$

Similarly, if  $P(A)$  denotes the pure state space of  $A$ , we have  $\|\xi\| = \sup_{\omega \in P(A)} \| |\xi|_\omega \|_2$ .

**Lemma 6.1** Assume  $A$  is commutative and  $\omega$  is a pure state of  $A$ . Let  $f \in C_c(\Sigma)$  and assume it takes values in  $A^\alpha = \{a \in A \mid \alpha_g(a) = a \text{ for all } g \in G\}$ .

Then, for all  $\xi \in H_A^G$ , we have

$$\| |\Lambda(f)\xi|_\omega \|_2 \leq \| |f|_\omega * |\xi|_\omega \|_2.$$

*Proof* Set  $E = \text{supp}(f)$  and let  $h \in G$ . Note that the assumption on  $f$  implies that  $\alpha_h^{-1}(f(g)) = f(g)$  for all  $g \in E$ . Moreover, note that  $\|u\|_\omega = 1$  for any  $u \in \mathcal{U}(A)$ . Using the triangle inequality for  $\|\cdot\|_\omega$  and the fact that  $\omega$  is multiplicative, we then get

$$\begin{aligned} |\Lambda(f)\xi|_\omega(h) &= \left\| \sum_{g \in E} \alpha_h^{-1}(f(g)) \alpha_h^{-1}(\sigma(g, g^{-1}h)) \xi(g^{-1}h) \right\|_\omega \\ &\leq \sum_{g \in E} \|f(g)\|_\omega \|\alpha_h^{-1}(\sigma(g, g^{-1}h))\|_\omega \|\xi(g^{-1}h)\|_\omega \\ &= \sum_{g \in E} \|f(g)\|_\omega \|\xi(g^{-1}h)\|_\omega = \sum_{g \in E} |f|_\omega(g) |\xi|_\omega(g^{-1}h) \\ &= (|f|_\omega * |\xi|_\omega)(h), \end{aligned}$$

and the desired inequality follows immediately.  $\square$

It is conceivable that Lemma 6.1 holds without having to assume that  $f$  takes values in  $A^\alpha$  if its conclusion is changed to: “For all  $\xi \in H_A^G$ , we have

$$\| |\Lambda(f)\xi|_\omega \|_2 \leq \| |f^\alpha|_\omega * |\xi|_\omega \|_2,$$



where  $f^\alpha$  is defined by  $f^\alpha(g) = \alpha_g^{-1}(f(g))$ , for  $g \in G$ ." Unfortunately, we have so far not been able to establish this inequality. With such a more general result at hand, we would not have to assume that  $f$  takes values in  $A^\alpha$  in the next proposition, and our results in the sequel would all also be true for a non-trivial  $\alpha$ .

**Proposition 6.2** *Assume  $A$  is commutative and  $G$  is  $\kappa$ -decaying with decay constant  $C$  for some  $\kappa : G \rightarrow [1, \infty)$ . Let  $f \in C_c(\Sigma)$  and assume it takes values in  $A^\alpha$ . Then*

$$\|\Lambda_\Sigma(f)\| \leq C\|f\|_{\alpha,\kappa}.$$

*Proof* Let  $\xi \in H_A^G$ . Then, using Lemma 6.1 and the  $\kappa$ -decay of  $G$ , we get

$$\begin{aligned} \|\Lambda(f)\xi\| &= \sup_{\omega \in P(A)} \|\Lambda(f)\xi|_\omega\|_2 \\ &\leq \sup_{\omega \in P(A)} \|f|_\omega * |\xi|_\omega\|_2 \\ &\leq C \sup_{\omega \in P(A)} \|f|_\omega\|_{2,\kappa} \| |\xi|_\omega \|_2 \\ &= C \sup_{\omega \in P(A)} \|f\kappa|_\omega\|_2 \| |\xi|_\omega \|_2 \\ &\leq C\|f\kappa\| \|\xi\| = C\|f\|_{\alpha,\kappa} \|\xi\|, \end{aligned}$$

the final equality being due to the fact that  $f$  takes values in  $A^\alpha$ . This shows that  $\|\Lambda(f)\| \leq C\|f\|_{\alpha,\kappa}$ . As  $\|\Lambda_\Sigma(f)\| = \|\Lambda(f)\|$ , the assertion is proven.  $\square$

**Corollary 6.3** *Assume  $A$  is commutative,  $\alpha$  is trivial and  $G$  is  $\kappa$ -decaying for some  $\kappa : G \rightarrow [1, \infty)$ . Then  $\Sigma = (A, G, \text{id}, \sigma)$  is  $H_\kappa^\Sigma$ -decaying.*

*Proof* Since  $A^\alpha = A$  when  $\alpha$  is trivial, the result is an immediate consequence of Proposition 6.2.  $\square$

The following result generalizes [4, Theorem 3.13].

**Corollary 6.4** *Assume that  $A$  is commutative,  $G$  is countable and  $\alpha$  is trivial. Let  $K : G \rightarrow [0, \infty)$  be a proper function.*

*Assume that  $G$  has polynomial  $H$ -growth (w.r.t.  $K$ ). Then there exists some  $s_0 > 0$  such that  $\Sigma$  is  $H_\kappa^\Sigma$ -decaying, where  $\kappa = (1 + K)^{s_0}$ .*

*More generally, assume that  $G$  has subexponential  $H$ -growth (w.r.t.  $K$ ). Let  $r \in (0, 1)$  and set  $\kappa_r = r^{-K}$ . Then  $\Sigma$  is  $H_{\kappa_r}^\Sigma$ -decaying.*

*Proof* Assume that  $G$  has polynomial  $H$ -growth (w.r.t.  $K$ ). Then, according to [4, Theorem 3.13, part 1)], there exists some  $s_0 > 0$  such that  $G$  is  $\kappa$ -decaying, where  $\kappa = (1 + K)^{s_0}$ . So the first statement follows from Corollary 6.3. The second statement is proven in the same way, using now [4, Theorem 3.13, part 2)].  $\square$

Assume  $A$  is commutative,  $G$  is countable and  $\alpha$  is trivial. In the setting of Corollary 6.4, the first assertion implies that if  $G$  has polynomial  $H$ -growth (w.r.t.  $K$ ) and we set  $\kappa_s = (1 + K)^s$  for  $s > 0$ , then the Fréchet space  $\bigcap_{s>0} H_{\kappa_s}^\Sigma$  (w.r.t. the obvious family

of seminorms) embeds (densely) in  $C_r^*(\Sigma)$ . We tend to believe that this also should hold when  $\alpha$  is non-trivial.

When  $A$  is commutative,  $\sigma$  is scalar-valued and  $\alpha$  is not assumed to be trivial, such a Fréchet space has been considered by Ji and Schweitzer [31] in the more general setting of actions by locally compact groups. By following a rather different method, involving a certain generalized Roe algebra, they show that if  $G$  has the so-called strong rapid decay (SRD) property (w.r.t. to some proper length function on  $G$ ), then the associated Fréchet space embeds as a *spectral invariant* dense  $*$ -subalgebra of  $C_r^*(\Sigma)$ . Moreover, it is shown in [10] that the converse statement is also true for discrete groups, and that, a discrete group  $G$  has property (SRD) if and only if  $G$  has polynomial growth in the usual sense (w.r.t. some proper length function).

We also include a result in the vein of [4, Theorem 5.9 and Corollary 5.15]:

**Corollary 6.5** *Assume  $A$  is commutative,  $G$  is countable with the Haagerup property and  $\alpha$  is trivial. Let  $K : G \rightarrow [0, \infty)$  be a Haagerup function for  $G$  (so  $K$  is negative definite and proper) and assume that  $G$  has subexponential  $H$ -growth (w.r.t.  $K$ ).*

*For each  $r \in (0, 1)$ , set  $\psi_r = r^K$  and  $L^r = L^{\psi_r}$ .*

*Then  $\{L^r\}_{r \rightarrow 1^-}$  is a bounded Fourier summing net for  $\Sigma = (A, G, \text{id}, \sigma)$ . Moreover,  $\Sigma$  has the bounded Fejér property.*

*Proof* Since each  $\psi_r$  is a normalized positive definite function on  $G$ , we know from Corollary 4.3 that  $\|L^r\| = 1$  for all  $r$ , so  $\{L^r\}_{r \rightarrow 1^-}$  is bounded. It is also evident that  $\psi_r$  converges pointwise to 1 when  $r \rightarrow 1^-$ .

For each  $r \in (0, 1)$ , set  $\kappa_r = r^{-K}$ . Corollary 6.4 gives that  $\Sigma$  is  $H_{\kappa_r}^\Sigma$ -decaying. As  $\psi_r \kappa_r = 1$ ,  $\psi_r \in \ell_{\kappa_r}^\infty(G, A)$  for all  $r$ . Hence, we may apply Proposition 5.11 to obtain the first assertion.

For each  $r \in (0, 1)$ , set  $\kappa'_r = r^{-K/2} = \kappa_{\sqrt{r}}$ . Then, for each  $r$ ,  $\Sigma$  is  $H_{\kappa'_r}^\Sigma$ -decaying and  $\psi_r \kappa'_r \in c_0(G, A)$ . We may therefore apply Proposition 5.12 and obtain the second assertion.

Finally, as an application of Corollary 6.5, we give a simple example illustrating how our work may be used to determine the ideal structure of certain group  $C^*$ -algebras.

**Example 6.6** Consider  $\Gamma = SL(2, \mathbb{Z})$  and let  $\lambda$  denote its left regular representation on  $\ell^2(\Gamma)$ . Denote the identity element in  $\Gamma$  by  $I_2$  and set  $S = \lambda(-I_2)$ . As  $S = S^* = S^{-1}$  is central in  $B = C_r^*(\Gamma)$ , we have

$$B = Bp \oplus Bq$$

where  $p, q$  are the central projections in  $B$  given by  $p = \frac{I+S}{2}, q = \frac{I-S}{2}$ .

In particular,  $B$  has at least two non-trivial ideals, namely  $Bp$  and  $Bq$ . In fact, these are the only non-trivial ideals of  $B$ . To see this, we may argue as follows.

Let  $Z = \{\pm I_2\}$  denote the center of  $\Gamma$  and set  $G = \Gamma/Z$ , so  $G$  is the modular group  $PSL(2, \mathbb{Z}) \simeq \mathbb{Z}_2 * \mathbb{Z}_3$ . Using [3, Theorem 2.1], we may write  $B \simeq C_r^*(A, G, \text{id}, \sigma)$  where  $A = C_r^*(Z) \simeq \mathbb{C}^2$  and  $\sigma : G \times G \rightarrow \mathcal{U}(A) \simeq \mathbb{T}^2$  is a suitably chosen cocycle. Now, as is well known,  $G \simeq \mathbb{Z}_2 * \mathbb{Z}_3$  is a Powers group [29] and  $G$  has the Haagerup property, the “block” length function  $K$  on  $G$  being a Haagerup function (see for

instance [6]). Moreover,  $G$  has polynomial H-growth, and therefore subexponential H-growth (w.r.t.  $K$ ), see [4, Example 3.12, part 4)]. Hence, Corollary 6.5 applies, showing the existence of a Fourier summing net for  $(A, G, \text{id}, \sigma)$ , which obviously preserves ideals of  $A$ . We can therefore conclude from Example 5.5 that there is a bijection between ideals of  $A \simeq \mathbb{C}^2$  and ideals of  $B$ , hence that  $B$  has exactly two non-trivial ideals, as desired.

Note that  $G$  is known to be exact (cf. [9, Theorem 5.2.7]), so one can avoid showing the existence of a Fourier summing net as we did above. However, we hope that the technique of proof might help to handle more complicated cases in the future.

**Acknowledgments** Most of the present work has been done during several visits made by R.C. at the Institute of Mathematics, University of Oslo in the period 2009–2013. He thanks the operator algebra group for their kind hospitality and the Norwegian Research Council for partial financial support. Both authors are grateful to the referees for their useful comments and for pointing out several misprints in the original manuscript. Erik Bédos was partially supported by the Norwegian Research Council.

## References

1. Anantharaman-Delaroche, C.: Systèmes dynamiques non commutatifs et moyennabilité. *Math. Ann.* **279**, 297–315 (1987)
2. Anantharaman-Delaroche, C.: Amenability and exactness for dynamical systems and their  $C^*$ -algebras. *Trans. Am. Math. Soc.* **354**, 4153–4178 (2002)
3. Bédos, E.: Discrete groups and simple  $C^*$ -algebras. *Math. Proc. Camb. Philos. Soc.* **109**, 521–537 (1991)
4. Bédos, E., Conti, R.: On twisted Fourier analysis and convergence of Fourier series on discrete groups. *J. Fourier Anal. Appl.* **15**, 336–365 (2009)
5. Bédos, E., Conti, R.: On discrete twisted  $C^*$ -dynamical systems, Hilbert  $C^*$ -modules and regularity. *Münster J. Math.* **5**, 183–208 (2012)
6. Bożejko, M.: Positive-definite kernels, length functions on groups and a noncommutative von Neumann inequality. *Studia Math.* **XCIV**, 107–118 (1989)
7. Bożejko, M., Fendler, G.: Herz–Schur multipliers and completely bounded multipliers of the Fourier algebra of a locally compact groups. *Boll. Un. Mat. Ital.* **3–A**, 297–302 (1984)
8. Bożejko, M., Fendler, G.: Herz–Schur multipliers and uniformly bounded representations of discrete groups. *Arch. Math. (Basel)* **57**, 290–298 (1991)
9. Brown, N.P., Ozawa, N.:  $C^*$ -Algebras and Finite-Dimensional Approximations. Graduate Studies in Mathematics, vol. 88. American Mathematical Society, Providence (2008)
10. Chen, X., Wei, S.: Spectral invariant subalgebras of reduced crossed product  $C^*$ -algebras. *J. Funct. Anal.* **197**, 228–246 (2003)
11. Cheriş, P.-A., Cowling, M., Jolissaint, P., Julg, P., Valette, A.: Groups with the Haagerup property. Gromov’s a-T-menability. *Progress in Mathematics*, vol. 197. Birkhäuser Verlag, Basel (2001)
12. de Cannière, J., Haagerup, U.: Multipliers of the Fourier algebras of some simple Lie groups and their discrete subgroups. *Am. J. Math.* **107**, 455–500 (1985)
13. Davidson, K.:  $C^*$ -algebras by Example. Fields Institute Monographs, vol. 6. American Mathematical Society, Providence (1996)
14. Dixmier, J.: *Topologie Générale*. PUF, Paris (1981)
15. Echterhoff, S.: Crossed products, the Mackey–Rieffel–Green machine and applications. Preprint (2010), [arXiv:1006.4975](https://arxiv.org/abs/1006.4975)
16. Echterhoff, S., Kaliszewski, S., Quigg, J., Raeburn, I.: Naturality and induced representations. *Bull. Aust. Math. Soc.* **61**, 415–438 (2000)
17. Echterhoff, S., Kaliszewski, S., Quigg, J., Raeburn, I.: A categorical approach to imprimitivity theorems for  $C^*$ -dynamical systems. *Mem. Am. Math. Soc.* **850**, 1–169 (2006)
18. Echterhoff, S., Williams, D.P.: Central twisted transformation groups and group  $C^*$ -algebras of central group extensions. *Indiana Univ. Math. J.* **51**, 1277–1304 (2002)
19. Exel, R.: Amenability for Fell bundles. *J. Reine Angew. Math.* **492**, 41–73 (1997)

20. Exel, R.: Exact groups, induced ideals and Fell bundles. Preprint (2000), [arXiv:math/0012091](https://arxiv.org/abs/math/0012091)
21. Exel, R.: Exact groups and Fell bundles. *Math. Ann.* **323**, 259–266 (2002)
22. Exel, R., Laca, M.: Continuous Fell bundles associated to measurable twisted actions. *Proc. Am. Math. Soc.* **125**, 795–799 (1997)
23. Exel, R., Ng, C.K.: Approximation property of  $C^*$ -algebraic bundles. *Math. Proc. Camb. Philos. Soc.* **132**, 509–522 (2002)
24. Gootman, E.C., Lazar, A.: Applications of noncommutative duality to crossed product  $C^*$ -algebras determined by an action or coaction. *Proc. Lond. Math. Soc.* **59**, 593–624 (1989)
25. Haagerup, U.: On the dual weights for crossed products of von Neumann algebras. II. Application of operator-valued weights. *Math. Scand.* **43** (1978/79), 119–140
26. Haagerup, U.: An example of a nonnuclear  $C^*$ -algebra, which has the metric approximation property. *Invent. Math.* **50** (1978/79), 279–293
27. Haagerup, U., Kraus, J.: Approximation properties for group  $C^*$ -algebras and group von Neumann algebras. *Trans. Am. Math. Soc.* **344**, 667–699 (1994)
28. Haagerup, U., de Laat, T.: Simple Lie groups without the approximation property. *Duke Math. J.* **162**, 925–964 (2013)
29. de la Harpe, P.: On simplicity of reduced group  $C^*$ -algebras. *Bull. Lond. Math. Soc.* **39**, 1–26 (2007)
30. de la Harpe, P., Skandalis, G.: Powers' property and simple  $C^*$ -algebras. *Math. Ann.* **273**, 241–250 (1986)
31. Ji, R., Schweitzer, L.B.: Spectral invariance of smooth crossed products, and rapid decay locally compact groups. *K-Theory* **10**, 283–305 (1996)
32. Jolissaint, P.: Rapidly decreasing functions in reduced  $C^*$ -algebras of groups. *Trans. Am. Math. Soc.* **317**, 167–196 (1990)
33. Jolissaint, P.: A characterization of completely bounded multipliers of Fourier algebras. *Colloq. Math.* **63**, 311–313 (1992)
34. Lafforgue, V., de la Salle, M.: Noncommutative  $L^p$  spaces without the completely bounded approximation property. *Duke Math. J.* **160**, 71–116 (2011)
35. Lance, E.C.: Hilbert  $C^*$ -modules. A toolkit for operator algebraists. London Mathematical Society Lecture Note Series, vol. 210. Cambridge University Press, Cambridge (1995)
36. Landstad, M.B., Phillips, J., Raeburn, I., Sutherland, C.E.: Representations of crossed products by coactions and principal bundles. *Trans. Am. Math. Soc.* **299**, 747–784 (1987)
37. Ozawa, N.: Weak amenability of hyperbolic groups. *Groups Geom. Dyn.* **2**, 271–280 (2008)
38. Packer, J.A., Raeburn, I.: Twisted crossed product of  $C^*$ -algebras. *Math. Proc. Camb. Philos. Soc.* **106**, 293–311 (1989)
39. Packer, J.A., Raeburn, I.: Twisted crossed product of  $C^*$ -algebras II. *Math. Ann.* **287**, 595–612 (1990)
40. Paulsen, V.: Completely bounded maps and operator algebras. Cambridge University Press, Cambridge (2002)
41. Pisier, G.: Similarity Problems and Completely Bounded Maps, 2nd edn. Lecture Notes in Mathematics, vol. 1618. Springer, Berlin (2001)
42. Rørdam, M., Sierakowski, A.: Purely infinite  $C^*$ -algebras arising from crossed products. *Ergod. Theory Dyn. Syst.* **32**, 273–293 (2012)
43. Sierakowski, A.: The ideal structure of reduced crossed products. *Münster J. Math.* **3**, 237–262 (2010)
44. Steenstrup, T.: Herz–Schur multipliers and non uniformly bounded representations of locally compact groups. Preprint (2010), [arXiv:1001.0326](https://arxiv.org/abs/1001.0326)
45. Tomiyama, J.: Structure of ideals and isomorphisms of  $C^*$ -crossed products by single homeomorphism. *Tokyo J. Math.* **23**, 1–13 (2000)
46. Tomiyama, J.: Classification of ideals of homeomorphism  $C^*$ -algebras and quasidiagonality of their quotient algebras. *Acta Appl. Math.* **108**, 561–572 (2009)
47. Williams, D.P.: Crossed Products of  $C^*$ -Algebras. Mathematical Surveys and Monographs, vol. 134. American Mathematical Society, Providence (2007)
48. Zeller-Meier, G.: Produits croisés d'une  $C^*$ -algèbre par un groupe d'automorphismes. *J. Math. Pures Appl.* **47**, 101–239 (1968)